J. of Natural & Physical Sciences Vol. 14 2000 No. 1-2

ISSN 097-3799

खण्ड 14 अंक 1 – 2, 2000

Vol. 14 No. 1 – 2, 2000

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

JOURNAL OF NATURAL & PHYSICAL SCIENCES

गुरुकुल कांगड़ी विश्वविद्यालय, हरिद्वार

Gurukula Kangri Vishwavidyalaya, Hardwar

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका
Journal of Natural & Physical Sciences

Journal of Natural & Physical Sciences, Vol. 14 (2000) No. 1-2

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Printed at : Kiran Printing Press, Kankhal-Hardwar Phone : 415975

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Journal of Natural & Physical Sciences, Vol. 14 (1-2) (2000) 1-6

RECIPROCAL CONTINUITY AND COMMON FIXED POINTS

R. P. PANT*

(Received 28.02.1999)

ABSTRACT

The aim of this paper is to obtain a common fixed point theorem for four mappings by using minimal type commutativity and contractive conditions and a new continuity condition, and to establish a situation in which a collection of maps has a common fixed point which is a point of discontinuity. Besides compatible maps, the theorem also deals with noncompatible maps- a class of mappings for which fixed point theorems are hardly found in the literature.

Mathematics Subject Classification 1991 : 54 H 25

Key words and phrases : Fixed point, compatible maps,

INTRODUCTION

In 1986 Jungck [3] generalized the concept of weakly commuting mappings by defining compatible mappings. Since then the study of common fixed points of generalized contractions satisfying compatibility or some other commutativity condition has emerged as an area of vigorous research activity. The central question concerning common fixed points of generalized contractions may be formulated as : given the selfmaps A, B, S, T of a metric space (X,d) satisfying a contractive condition what assumptions on commutativity, continuity and contractive condition guarantee a common fixed point of $A, B, S,$ and T. For compatible maps satisfying the contractive condition

(1) $d(Ax,By) < M(x,y,) = \max \{d(Sx,Ty), d(Ax,Sx), d(By,Ty), [d(Ax,Ty) + d(By,Sx)]/2\}$

the minimal additional conditions on (1), known as of now , are those included in the following stronger forms of (1) [1-4, 6, 7, 9]

(2) $d(Ax, By) \leq k M(x,y)$, $o \leq k < 1$,

(3) $d(Ax,By) \leq \phi(M(x,y))$

where $\phi : R_+ \to R_+$ is an upper semicontinuous function such that $\phi(t) < t$ for each $t > o$, and

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(4) there exists a funcation $\delta:(0,\infty)\to(0,\infty)$, which is nondecreasing or lower semicontinuous such that

$$\varepsilon \le M(x,y) < \varepsilon + \delta(\varepsilon) \Rightarrow d(Ax, By) < \varepsilon$$

The common fixed point theorems involving any of the conditions (2), (3) or (4) assume atleast one of the mapping to be continuous, e. g. , in [1-4, 6,7,9]. It may be pointed out here that it is known since the paper of Kannan [5] in 1968 that there exist maps that have a discontinuity in their domain but which have fixed points . However, in every case the mappings involved were continuous at the fixed point. In a recent study [10], the present author introduced the notion of reciprocal continuity. In the present paper, using the notion of reciprocal continuity we establish a situation in which the fixed point is a point of discontinuity.

Two selfmaps A and S of a metric space (X,d) are called reciprocally continuous if $\lim_n ASx_n = At$ and $\lim_n SAx_n = St$ whenever $\{x_n\}$ is a sequence such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X. If A and S are both continuous then they are obviously reciprocally continuous but the converse is not true . We illustrate this in the following pages.

Two selfmaps A and S of a metric space (X,d) are called compatible if $\lim_n d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X. In 1994 , the present author [8] introduced the notion of R-weakly commuting mappings. Two selfmaps A and S of a metric (X,d) are defined to be R-weakly commuting at a point x in X if $d(ASx,SAx) \le R\, d(Ax,Sx)$ for some $R > 0$. The maps A and S are called pointwise R-weakly commuting on X if given x in X there exists $R > 0$ such that $d(ASx,SAx) \le R\, d(Ax,Sx)$ .The definition implies that pointwise R-weakly commuting maps commute at their coincidence points . The converse of this is also true . for, if A and S commute at their coincidence points , we can define $R = \max\{1, d(ASx,SAx)/d(Ax,Sx)\}$ when $Ax \ne Sx$ and $R = 1$ when $Ax = Sx$. Therefore, pointwise R-weakly commutativity is equivalent to commutativity at coincidence points . In other words, A and S can fail to be pointwise R-weakly commuting only if they possess a coincidence point at which they do not commute. Moreover, since contractive conditions exclude the possibility of simultaneous existence of a common fixed point and a coincidence point at which the mappings do not commute, it follows that pointwise R-weak commutativity is a necessary, hence minimal, condition for the existence of common fixed points of contractive type mapping pairs . This has been demonstrated in the following pages.

In the present paper, using the notions of pointwise R-weak commutativity and

reciprocal continuity we prove a common fixed point theorem for four mappings satisfying the contractive condition (1) and establish a situation in wich the common fixed point is a point of discontinuity. Our theorem is perhaps the first result to guarantee a comman fixed point under condition (1) and without assuming continuity of maps.

RESULTS

THEOREM. Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of selfmappings of a metric space (X,d) satisfying

(i) $AX \subset TX$, $BX \subset SX$

(ii) $d(Ax, By) < M(x,y)$ whenever $M(x,y) > 0$.

Suppose that one of the pairs (A,S) or (B,T) is compatible and the other is noncompatible. If the mappings in the compatible pair be reciprocally continuous then A, B, S and T have a unique common fixed point.

Proof. Let B and T be noncompatible mappings and A and S be reciprocally continuous compatible mappings. Then noncompatibility of B and T implies that there exists some sequence $\{x_n\}$ in X such that $\lim_n Bx_n = \lim_n Tx_n = t$ for some t in X while $\lim_n d(BTx_n, TBx_n)$ is either nonzero or nonexistent. Since $BX \subset SX$, corresponding to each x_n there exists a y_n in X such that $Bx_n = Sy_n$. Thus $Bx_n \to t$, $Tx_n \to t$ and $Sy_n \to t$. We claim that $Ay_n \to t$. If not, then there exists a subsequence $\{Ay_m\}$ of $\{Ay_n\}$, a number $r > 0$ and a positive integer M such that for each $m \geq M$ we have $d(Ay_m, t) \geq r$, $d(Ay_m, Bx_m) \geq r$ and $d(Ay_m, Bx_m) < \max\{d(Sy_m, Tx_m), d(Ay_m, Sy_m), d(Bx_m, Tx_m), [d(Ay_m, Tx_m) + d(Bx_m, Sy_m)]/2\}$

$$= \max\{d(Ay_m, Bx_m), d(Ay_m, Tx_m)/2\} < \max\{d(Ay_m, Bx_m), (d(Ay_m, Bx_m) + d(Bx_m, Tx_m))/2\}$$

$$= d(Ay_m, Bx_m),$$

a contradiction. Hence $\lim_n Ay_n = t$, $\lim Sy_n = t$, $\lim_n Bx_n = t$ and $\lim_n Tx_n = t$ where $Sy_n = Bx_n$. Since A and S are reciprocally continuous, we get $\lim_n SAy_n = St$ and $\lim_n ASy_n = At$. Compatibility of A and S implies that $\lim_n d(ASy_n, SAy_n) = 0$, that is, $d(At, St) = 0$. Thus $At = St$. Since $AX \subset TX$, there exists some point w in X such that $At = Tw$. Now if $Tw \neq Bw$, by (ii) we get

$$d(At, Bw) < \max\{d(St, Tw), d(At, St), d(Bw, Tw), [d(At, Tw) + d(Bw, St)]/2\}$$

$$= d(Bw, Tw) = (Bw, At),$$

a contradiction . Hence $Bw = Tw$ and $St = At = Tw = Bw$. Pointwise R- weak commutativity of B and T implies that there exists $R > 0$ such that $d(BTw, TBw) \leq R\, d(\mathrm{Bw}, Tw) = 0$, that is , $BTw = TBw$. Moreover , $BBw = BTw = TBw = TTw$. Similarly , compatibility of A and S implies that $ASt = SAt$ and $AAt = ASt = SAt = SSt$. Now if $At \neq AAt$, using (ii) we get

$$d(At, Bw) < \max\{d(St, Tw), d(At, St), d(Bw, Tw), [d(At, Tw) + d(Bw, St)]/2\}$$

$$= d(Bw, Tw) = d(Bw, At),$$

a contradiction, Hence $Bw = Tw$ and $St = At = Tw = Bw$. Pointwise R-weak commutativity of B and T implies that there exists $R>0$ such that $d(BTw, TBw) \leq R\, d(Bw, Tw) = 0$, that is, $BTw = TBw$, Morever, $BBw = BTw = TBw = TTw$. Similarly, compatibility of A and S implies that $ASt = SAt$ and $AAt = ASt = SAt = SSt$. Now if At $\neq AAt$, using (ii) we get

$$d(At, AAt) = d(AAt, Bw) < M(At, w) = d(AAt, Bw),$$

a contradiction . Hence $At = AAt = SAt$ and At is a common fixed point of A and S. Similarly $Bw(= At)$ is a common fixed point of B and T . Uniqueness of the common fixed point is a consequence of the contractive condition (ii) . The proof is similar when A and S are assumed noncompatible and B and T are assumed reciprocally continuous compatible mappings . This completes the proof of the theorem.

We now give an example to illustrate the above theorem.

EXAMPLE. Let $X = [2, 20]$ and d be the usual metric on X . Define A , B , S , T : $X \to X$ by

$$A2 = 2, \ Ax = 3 \text{ if } x > 2$$

$$Bx = 2 \text{ if } x = 2 \text{ or } \geq 5, \ Bx = 6 \text{ if } 2 < x \leq 5$$

$$S2 = 2, \ Sx = 6 \text{ if } x > 2$$

$$T2 = 2, \ Tx = 7 + x \text{ if } 2 < x \leq 5, \ Tx = x - 3 \text{ if } x > 5.$$

Then A , B , S and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$. It may be verified in this example that A and S are reciprocally continuous compatible mappings and that all the mappings involved are discontinuous at

the common fixed point. The mappings B and T are noncompatible but pointwise R-weakly commuting. B and T are pointwise commuting since they commute at their coincidence points. To see that B and T are noncompatible, consider a decreasing sequence $\{x_n\}$ in X such that $x_n = 5+1/n$. Then $Bx_n = 2$, $Tx_n \to 2$, $TBx_n = 2$ and $BTx_n \to 6$. Hence B and T are noncompatible. It may also be verified that A, B, S and T satisfy the contractive condition (1) but not (2), (3) or (4). If possible, suppose that condition (3) is satisfied. Take $x > 2$ and $y_n = 2+1/n$. Then $d(Ax, By_n) = 3$ and $M(x, y_n) = 3+1/n \to 3$, so by (3) we infer that

$$3 \le \lim_{M\to 3} \sup \phi\big(M(x,y_n)\big) = \lim_{t\to 3} \sup \phi(t) = \phi(3) < 3,$$

a contradiction. Therefore, (3) does not hold. Hence condition (4) is not satisfied either, because as shown in [2], condition (4) implies condition (3). Condition (2) is a particular case of condition (3).

REMARK. In this remark we demonstrate that pointwise R-weak commutativity is a necessary condition for the existence of common fixed points of contractive type mapping pairs. So, let us assume that the selfmappings A and S of a metric space (X,d) satisfy the contractive condition

$$d(Ax, Ay) < \max\{d(Sx, Sy), d(Ax, Sx), d(Ay, Sy), d(Ax, Sy), d(Ay, Sx)\},$$

which is one of the most general contractive conditions for a pair of selfmappings. If possible, suppose that A and S fail to be pointwise R-weakly commuting and yet have a common fixed point z.

Then $z = Az = Sz$ and there exists x in X such that $Ax = Sx$ but

$ASx \neq SAx$. clearly, $z \neq x$ and $Az \neq Ax$ since $ASz = SAz = z$. But then by virtue of the contractive condition we get

$$d(Ax, Az) < \max\{d(Sx, Sz), d(Ax, Sx), d(Az, Sz), d(Ax, Sz), d(Az, Sx)\}$$

$$= d(Ax, Az),$$

a contradiction. Hence pointwise R-weak commutativity is a necessary condition for the existence of common fixed points of contractive type mapping pairs.

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ON AN ABSTRACT VOLTERRA INTEGRODIFFERENTIAL EQUATION

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(Received 10.11.1999)

ABSTRACT

The objective of this paper is to study the existence, uniqueness and other properties of the solutions of an integrodifferential equation of the form

$$u'(t) + Au(t) = f(t, u(t)) + \int_0^t a(t,s) g(s, u(s)) ds,$$

$$u(0) = u_0.$$

The technique used in our analysis are based on the method of succesive approximations and integral inequalities recently established by Pachpatte.

INTRODUCTION

In the present paper, we consider the following abstract integrodifferenital equation

$$\left. \begin{aligned} u'(t)+Au(t)=f(t,u(t))+\int_0^t a(t,s)g(s,u(s))ds, \\ u(0)=u_0, \end{aligned} \right\} \tag{1.1}$$

Where $A : D(A) \subset X \to X$ and $-A$ is an infinitesimal generator of c_0-semigroup $T(t), t \geq 0$ on $X(D(A)$ is defined in the next section), the nonlinear functions f, g : $[0, \alpha] \times X \to X$ and the kernel a : $[0, \alpha] \times [0, \alpha] \to R$, are continuous, R denotes the set of real numbers and $\alpha > 0$, is some real number.

The equations of the type (1.1) or its special forms serve as models for various partial differential equations or partial integrodifferential equations arising in heat flow in material with memory, viscoelasity and reaction-diffusion problems, see [2-6,15,20] and some of the references given therein. Many authors have studied

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the existence, uniqueness, continuation and other properties of solutions of the special forms of the equations (1.1); (see, [1,6,8-11,14,20-22] by using various methods. Our work in the present paper is motivated by the results obtained by M.L. Heard [9] and is influenced by the work of Travis and Webb [22].

The main purpose of this paper is to study the existence, uniqueness, contiuation and other properties of the solutions of (1.1). The main tools employed in our analysis are based on the semigroup theory, the method of sucessive approximations and integral inequalities established by Pachpatte in [16].

The paper is organised as follows. In the next section, we represent the preliminaries and statements of our main results. Then we deal with proofs of the Theorems 1-3. Then we prove the Theorems 4-5. Finally, we give some examples to illustrate the applications of our Theorems 1 and 5.

STATEMENT OF THE RESULTS

Before we state our results, we give the following basic concepts and definitions used in our subsequent discussion :

Let X be a Banach space with the norm $\| \cdot \|$ and $-A$ is infinitesimal generator of c_0-semigroup T, on a Banach space. The set of bounded linear operators $\{T(t): t \in R_+\}$, Where $R_+=[0, \infty]$ is *a* c_0-semigroup on X if

(i) $T(t+s) = T(t)T(s)=T(s)T(t), s, \quad t \geq 0,$

(ii) $T(0) = I$ (the identity operator),

(iii) $T(.)$ is strongly continuous in $t \in R_+$

(iv) $\| T(t) \| \leq Me^{\omega t}$ for $M, \omega > 0, \ t \in R_+$,

The operator A is generator of $T(.)$ if

$$Ax = \lim_{h-o+}(T(h) - T(0)) / h)x \quad and \quad D(A),$$

the domain of A, is the set of $x \in X$ for which the limit exists , (see[1,pp.24 and 25]).

We also consider the following equations in our study

$$\left.\begin{aligned} &z'(t)+Az(t)=F(t,z(t))+\int_0^t a(t,s)G(s,v(s))ds, \\ &z(0)=z_0, \end{aligned}\right\} \tag{2.1}$$

where $F,G \in C([0,\alpha] \times X, X)$.

$$\left.\begin{aligned} &v'(t)+Av(t)=F(t,v(t))+\int_0^t a(t,s)g(s,v(s))ds+H(t,v(t)) \\ &v(0)=v_0, \end{aligned}\right\} \tag{2.2}$$

where the functions f,g,a and generator A are the same as defined in (1.1) and nonlinear continuous functions from $[0,\alpha] \times X$ into, X.

A continuous solution u of the integral equation

$$u(t)=T(t)u_0+\int_0^t T(t-s)f(s,u(s))ds+\int_0^t T(t-s)\int_0^s a(s,\tau)g(\tau,u(\tau))d\tau ds, \tag{2.3}$$

is called the mild solution of (1.1). Similarly, we can define the mild solutions of (2.1) and (2.2).

For convenience, we list the following hypotheses used in our subsequent discussion.

(H_1) f and g are the nonlinear functions from $R_+ \times X$ into X. Suppose that there are positive nondecreasing functions $h_i : [0,\infty) \to (0,\infty)$, $i=1,2$ such that

(i) $\|f(t,x_1)\| \le h_1(x)$,

(ii) $\|g(t,x_1)\| \le h_2(x)$,

(iii) $\|f(t,x_1)-f(t,x_2)\| \le h_1(x)\|x_1-x_2\|$,

(iv) $\|g(t,x_1)-g(t,x_2)\| \le h_2(x)\|x_1-x_2\|$,

if $\|x_1\|,\|x_2\| \le x$ and $t \in [0,\alpha]$, for some $\alpha > 0$.

(H_2) $a(t,s) : [0,\alpha] \times [0,\alpha] \to R$ is a continuous function and satisfies the uniform Hölder continuity condition in both the variables with exponent ρ i.e. there exists a positive constant $b_0>0$ such that

$$\left|a(t_1,s_1)-a(t_2,s_2)\right| \le b_0\left(\left|t_1-t_2\right|^{\rho}+\left|s_1-s_2\right|^{\rho}\right),$$

for all $t_1,t_2,s_1,s_2 \in [0,\alpha]$.

(H_3) f and g are nonlinear continuous functions from $R_+ \times X$ into X. Suppose that there are positive nondecreasing functions $h_i : [0,\infty) \to (0,\infty)$, $i=1,2$, such that

(i) $\|f(t,x) \le h_1(t_1)$,

(ii) $\|g(t,x)\| \le h_2(t_1)$,

(iii) $\|f(t,x_1)-f(t,x_2)\| \le h_1(t_1)\|x_1-x_2\|$,

(iv) $\|g(t,x_1)-g(t,x_2)\| \le h_2(t_1)\|x_1-x_2\|, \qquad t_1 \ge t \ge 0.$

(H_4) H is a nonlinear continuous function from $R_+ \times X$ into X. There is a nonnegative constant L, such that

$$\|H(t,x)\| \le L \in \|x\|,$$

where $\in > 0$ is an arbitrary constant and $t \in [0,\alpha]$.

We need the following Lemmas in our subsequent discussion;

Lemma 1 (Webb [22 pp. 295-303]). Let $T(t), t \ge 0$ be a strongly continuously semigroup of linear operators with infinitesimal generator A. Let k_1 maps $[0,t_1]$ into X continuously and let

$$q(t)=\int_0^t T(t-s)k_1(s)ds, \quad 0 \le t \le t_1.$$

If k_1 is continuously differentiable, then q is continuously differentiable and for

$0 \le t \le t_1, q(t) \in D(A)$ and

$$q'(t) = Aq(t) + k_1(t) = T(t)k(0) + \int_0^t T(t-s)k_1'(s)ds.$$

Lemma 2 (Pachpatte [16,pp.794-802]). Let $x(t)$, $p(t)$ and $r(t)$ be real valued nonnegative continuous function defined on $I = [0,\infty]$, for which the inequality

$$x(t) \le x_0 + \int_0^t p(s)x(s)ds + \int_0^t p(s)\left(\int_0^s r(\tau)x(\tau)d\tau\right)ds, \quad t \in I$$

holds, where x_0 is a nonnegative constant then

$$x(t) \le x_0(1 + \int_0^t p(s)\exp(\int_0^s (p(\tau) + r(\tau))d\tau)ds),$$

for $t \in I$.

The following Lemma which is slight variant of Lemma 2 is also useful in our discussion.

Lemma 3 (Pachpatte [16,pp. 794-802]). Let $x(t)$, $p(t)$ and $r(t)$ be real valued non-negative continuous functions defined on $I = [0,\infty)$ and $n(t)$ be a positive, monotonic nondecrasing continuous function defined on I, for which the inequality

$$x(t) \le n(t) + \int_0^t p(s)x(s)ds + \int_0^t p(s)(\int_0^s r(\tau)x(\tau)d\tau)ds, \quad t \in I.$$

holds. Then

$$x(t) \le n(t)(1 + \int_0^t p(s)\exp(\int_0^s (p(\tau) + r(\tau))d\tau)ds), \ t \in I.$$

We are now in position to state our main results to be proved in this paper.

Theorem 1. Let hypotheses (H_1) and (H_2) hold then for every $u_0 \in X$ the initial

value problem (1.1) has unique mild solution $u(t) \in X$ on the interval $t \in [0,c]$ for some c>0.

Theorem 2. Suppose that the functions f and g satisfy (H_3) and $a(t,s)$ satisfies (H_2). Then for every $u_0 \in X$ the initial value problem (1.1) has a unique mild solution $u(t) \in X$ on R_+.

We next prove the continuation property of the solution of (2.3).

Theorem 3. Suppose that the hypothesis (H_2) is satisfied. Also assume that for each $\alpha > 0$ there exist positive nondecreasing functions $h_{i\alpha} : [0,\infty) \to (0,\infty)$, i=1,2, such that (H_1) is satisfied. Let $\alpha_0 > 0$ be such that there exists a mild solution u of (1.1) on $[0,\alpha_0)$ but that cannot be continued beyond $[0,\alpha_0]$. Then either $\alpha_0 = +\infty$ or

$$\limsup_{t \to \alpha_0} \|u(t)\| = +\infty.$$

Remark 1. We note that the solution of (2.3) is the mild solution of the equation (1.1). The equation (1.1) when $A = A(t)$ (A depends upon t) is a negative generator of an analytic semigroup with different assumptions on $A(t)$, has been investigated by Heard [9], by using Sobolevsky-Tanabe theory and Banach fixed point theorem. Also, Webb [22], has studied (1.1) when A is linear operator and generates an analytic semigroup with different conditions on the functions f and g and generator A, by using contraction mapping theorem . Here our approach to the problem is different from those of [9,22].

In the following theorem we establish the continuous dependence of the solutions of (1.1) on the functions and initial values involved in (1.1).

Theorem 4. Assume that the hypotheses (H_1) - (H_2) hold. Also assume that for an arbitrary $\varepsilon > 0$ the following conditions are satisfied

$$\|u_0 - z_0\| \leq \varepsilon, \tag{2.4}$$

$$\|f(t,u(t)) - F(t,z(t))\| \leq \varepsilon, \tag{2.5}$$

and

$$\|g(t,u(t)) - G(t,z(t))\| \leq \varepsilon, \tag{2.6}$$

where $t \in [0,\alpha]$ and $u(t), z(t) \in X$. Then the mild solutions of (1.1) depend continuously on the functions and initial values involved in (1.1) defined on $[0,\alpha]$.

Remark 2. It is to be noted that the continuous dependence of solutions of (1.1) when $A=A(t)$ (depends upon t) is a negative generator of analytic semigroup has been obtained by Heard [9], by constructing a corresponding sequence of integrodifferential equations. Here our technique and conditions on the functions involved in (1.1) are different from those of [9].

Now we shall investigate the behavioural relationships between the solutions of (1.1) and (2.2) in the following

Theorem 5. Assume that the hypotheses (H_1), (H_2) and (H_4) hold. Also assume that for an arbitrary $\varepsilon > 0$, the condition

$$\|u_0 - v_0\| < \varepsilon, \tag{2.7}$$

satisfies and the solution of (1.1) is bounded. Then there is an equivalence relationship between the mild solutions of (1.1) and (2.2).

Remark 3. Theorem 5 establishes the behavioural relationship between the solutions of (1.1) and (2.2). In the case where $A= A(t)$ (depends upon t) is one parameter family of closed linear operators then the behavioural relationship has studied by Pachpatte [17-19] by using the different conditions on nonlinear functions and operator $A(t)$. Our method and condtions on nonlinear functions f,g and generator A are entirely different from those used in [17-19].

PROOFS OF THEOREMS 1-3

A local solution of equation (2.3) will be shown to exist by successive iteration as follows.

First, we put

$$u_0(t) = T(t)u_0 \tag{3.1}$$

and define $u_{j+1}(t)$ by

$$u_{j+1}(t) = u_0(t) + \int_0^t T(t-s) f\big(s, u_j(s)\big) ds + \int_0^t T(t-s) \int_0^s a(s,\tau) g\big(\tau, u_j(\tau)\big) d\tau ds, \tag{3.2}$$

From hypothesis (H_2), we observe that

$$|a(t,s)| \le |a(t,s) - a(0,0)| + |a(0,0)|$$

$$\le b_0(2\alpha) + |a(0,0)|,$$

$$= \beta \tag{3.3}$$

where $\beta = 2b_0\alpha + |a(0,0)|$,

For $0 \le t \le \alpha_0$, *let* $\|T(t)\| \le M$, for some $M > 0$ and $\alpha_0 = \alpha$ (α as in hypothesis). (H_1)). Let $k > 2M\|u_0\|$. From (3.1) and above assumptions, we get,

$$\|u_0(t)\| \le \|T(t)u_o\| \le \|T(t)\|\|u_0\| \le M\|u_0\| \le k/2 \tag{3.4}$$

for $0 \le t \le c(k) = c$ (and $0 < c \le \alpha_0$)

Let

$$K = \frac{k}{M[2h_1(k) + \beta h_2(k)c]}$$

If $K \ge c$, we redefine $h_i, i = 1,2$ so that (H_1) still holds and $K \le C$,

From (3.2) - (3.4),conditions (i)-(ii) of hypothesis (H_1), hypothesis (H_2) and $\|T(*)\| \le M$, we obtain

$$\|u_{n+1}(t)\| \le \|u_0(t)\| + \int_0^t \|T(t-s)\|\|f(s,u_n(s))\|ds + \int_0^t \|T(t-s)\|\int_0^s |a(s,\tau)|\|g(\tau,u_n(\tau))\|d\tau ds$$

$$\le k/2 + \int_0^t Mh_1(k)ds + \int_0^t M\int_0^s \beta h_2(k)d\tau ds$$

$$\le k/2 + \frac{Mt}{2}[2h_1(k) + \beta h_2(k)c] = k \tag{3.5}$$

for $0 \le t \le K$. Therefore $\|u_n(t)\| \le k$, for all n and $t \in [0,K]$

Taking $j = 0,1$ in (3.2) and using (3.3) conditions (i) - (ii) of hypothesis (H_1), hypothesis (H_2) and (3.5), we have

$$\|u_2(t)-u_1(t)\| \le \int_0^t \|T(t-s)\| h_1(k)\|u_1(s)-u_0(s)\|ds + \int_0^t \|T(t-s)\|\int_0^s \beta h_2(k)\|u_1(\tau)-u_0(\tau)\|d\tau ds$$

$$\le \int_0^t Mh_1(k)2kds + \int_0^t M\int_0^s \beta h_2(k)2kd\tau ds$$

$$\le \frac{2Mtk}{2}\left[2h_1(k)+\beta h_2(k)c\right]. \tag{3.6}$$

Assume for $j = 1,2,\ldots\ldots, n\text{-}1$ and $t \in [0,K]$ that

$$\|u_j(t)-u_{j-1}(t)\| \le \left[\frac{Mt}{2}\{2h_1(k)+\beta h_2(k)c\}\right]^{j-1} \frac{2k}{(j-1)!} \tag{3.7}$$

Then, from (3.1) - (3.3), conditions (iii) - (iv) of hypothesis (H_1), hypothesis (H_2) and (3.7), we get

$$\|u_{n+1}(t)-u_n(t)\| \le \int_0^t M\|f(s,u_n(s))-f(s,u_{n-1}(s))\|ds$$

$$+\int_0^t M\int_0^s \beta\|g(\tau,u_n(\tau))-g(\tau,u_{n-1}(\tau))\|d\tau ds$$

$$\le \int_0^t Mh_1(k)\|u_n(s)-u_{n-1}(s)\|ds + \int_0^t M\int_0^s \beta h_2(k)\|u_n(\tau)-u_{n-1}(\tau)\|d\tau ds$$

$$\le \int_0^t Mh_1(k)\left[\frac{M}{2}\{2h_1(k)+\beta h_2(k)c\}\right]^{n-1} \frac{s^{n-1}2kds}{(n-1)!}$$

$$+\int_0^t M\beta h_2(k)\int_0^s\left[\frac{M}{2}2h_1(k)+\beta h_2(k)c\right]^{n-1}\frac{\tau^{n-1}}{(n-1)!}2kd\tau ds$$

$$\leq\left[\frac{Mt}{2}\{2h_1(k)+\beta h_2(k)c\}\right]^n\frac{2k}{n!} \tag{3.8}$$

for $t\in[0,K]$. Hence $\{u_j\}$ is strongly convergent and converges to a funtion u uniformly on $t\in[0,K]$ by letting $j\to\infty$ in (3.2), we see that u satisfies (2.3) on $[0,K]$.

Let $v(t)$ be another solution of (2.3) and $\|v(t)\|\leq k';k'\leq\alpha$ (α is as in hypothesis H_1). Then

$$\|u(t)-v(t)\|\leq\int_0^t M\|f(s,u(s))-f(s,v(s))\|ds$$

$$+\int_0^t M\int_0^s\beta\|g(\tau,u(\tau))-g(\tau,v(\tau))\|d\tau ds$$

$$\leq\in+\int_0^t Mh_1(k")\|u(s)-v(s)\|ds$$

$$+\int_0^t Mh_1(k")\int_0^s\beta\frac{h_2(k")}{h_1(k")}\|u(\tau)-v(\tau)\|d\tau ds, \tag{3.9}$$

where $\varepsilon>0$ is an arbitrary and $k''=\max\{k,k'\}$. Now, using Lemma 2, we obtain

$$\|u(t)-v(t)\|\leq\varepsilon\left[1+\int_0^t Mh_1(k")\exp\left(\int_0^s\left[Mh_1(k")+\beta\frac{h_2(k")}{h_1(k")}\right]d\tau ds\right)\right] \tag{3.10}$$

Since $\in$ is an arbitrary constant and therefore (3.10) gives the uniqueness of the solution of (1.1). This completes the proof of Theorem 1.

We define u_j again by (3.1) - (3.2). For $0 \leq t \leq \alpha$, using the conditions (i) - (iv) of hypothesis (H_3), hypothesis (H_2) and (3.1) - (3.4), we have

$$\|u_1(t) - u_0(t)\| \leq \int_0^t \|T(t-s)\| \, \|f(s, u_0(s)) - f(s,0)\| ds$$

$$+\int_0^t \|T(t-s)\| \int_0^s |a(s,\tau)| \|g(\tau, u_0) - g(\tau, 0)\| d\tau ds$$

$$+\int_0^t \|T(t-s)\| \left\{ \|f(s,0)\| + \int_0^s |a(s,\tau)| \|g(\tau,0)\| d\tau \right\} ds$$

$$\leq \int_0^t Mh_1(\alpha)\|u_0(s)\| ds + \int_0^t M \int_0^s \beta h_2(\alpha) \|u_0(\tau)\| d\tau ds + \int_0^t M \left\{ \|f(s,0)\| + \int_0^s \beta \|g(\tau,0)\| d\tau \right\} ds$$

$$\leq Mh_1(\alpha)\frac{kt}{2} + M\beta h_2(\alpha)\frac{t^2}{2} k/2 + M\{h_1(\alpha) + \beta h_2(\alpha) t/2!\} t$$

$$\leq Mth(\alpha)C_1 \tag{3.11}$$

where $h(\alpha) = (h_1(\alpha) + \beta h_2(\alpha)\,\alpha)$ and $C_1 = k/2+1$. Now, from (3.1)- (3.4), conditions (iii)-(iv) of (H_3) and (3.11), we obtain

$$\|u_2(t) - u_1(t)\| \leq \int_0^t Mh_1(\alpha)\|u_1(s) - u_0(s)\| ds + \int_0^t M \int_0^s \beta h_2(\alpha) \|u_1(\tau) - u_0(\tau)\| d\tau ds$$

$$\leq \int_0^t Mh_1(\alpha) Mh(\alpha) c_1 s ds + \int_0^t M\beta h_2(\alpha) \int_0^s Mh(\alpha) c_1 \tau d\tau ds$$

$$\leq (Mh(\alpha)t)^2 c_1 / 2! \tag{3.12}$$

Assume that

$$\|u_j(t) - u_{j-1}(t)\| \leq (Mh(\alpha)t)^j c_1 / j!, \tag{3.13}$$

for $j=1,2,\ldots.n$. Then by using (3.1)- (3.3), conditions (iii)-(iv) of hypothesis (H_3) and (3.13) we have

$$\|u_{n+1}(t)-u_n(t)\| \le \int_0^t Mh_1(\alpha)\|u_n(s)-u_{n-1}(s)\|ds + \int_0^t M\int_0^s \beta h_2(\alpha)\|u_n(\tau)-u_{n-1}(\tau)\|d\tau ds$$

$$\le \int_0^t Mh_1(\alpha)\left(Mh(\alpha)\right)^n \frac{c_1}{n!} s^n ds + \int_0^t M\int_0^s \beta h_2(\alpha)\left(Mh(\alpha)\right)^n \frac{c_1}{n!}\tau^n d\tau ds$$

$$\le M\left(Mh(\alpha)\right)^n c_1 \frac{t^n+1}{(n+1)!}\left[h_1(\alpha)+\beta h_2(\alpha)\alpha\right]$$

$$=\left(Mh(\alpha)t\right)^{n+1} c_1/(n+1)!, \tag{3.14}$$

completing the induction. Therefore $\{u_j(t)\}$ is strongly convergent on $[0,\alpha]$. Since α is arbitrary, $\{u_j(t)\}$ converges on $[0,\infty)$ to a solution of (2.3), uniformly on the compact subsets. Uniqueness and continuity follows from Theorem1. The proof of Theorem 2 is complete.

Let $\alpha_0=\alpha$ and suppose that $\alpha<\infty$ and $\|u(t)\|\le M_1$ for $0\le t\le\alpha$. We shall first show that $st-\lim\limits_{t\to\alpha} u(t)(\equiv u(\alpha))$ exist

Let

$$v(t)=\int_0^t T(t-s)f(s,u(s))ds+\int_0^s a(s,\tau)g(\tau,u(\tau))d\tau ds. \tag{3.15}$$

For $0\le t_0<t<\infty$,

$$v(t)-v\left(\hat{t}\right)=\int_0^t\left(T(t-s)-T\left(\hat{t}-s\right)\right)\left\{f(s,u(s))+\int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\right\}ds$$

$$+\int_{\hat{t}}^{t} T\left(\hat{t}-s\right)\left\{f(s,u(s))+\int_{0}^{s} a(s,\tau)g(\tau,u(\tau))d\tau\right\}ds$$

$$=\int_{0}^{t} T(t-s)\left(I-T\left(\hat{t}-t\right)\right)\left\{f(s,u(s))+\int_{0}^{s} a(s,\tau)g(\tau,u(\tau))d\tau\right\}ds$$

$$\times\int_{\hat{t}}^{t} T(t-s)\left\{f(s,u(s))+\int_{0}^{s} a(s,\tau)g(\tau,u(\tau))d\tau\right\}ds$$

$$\equiv J_1+J_2 \tag{3.16}$$

where

$$J_1=\int_{0}^{t} T(t-s)\left(I-T\left(\hat{t}-t\right)\right)\left\{f(s,u(s))+\int_{0}^{s} a(s,\tau)g(\tau,u(\tau))d\tau\right\}ds \tag{3.17}$$

and

$$J_2=\int_{\hat{t}}^{t} T(\hat{t}-s)\left\{f(s,u(s))\int_{0}^{s} a(s,\tau)g(\tau,u(\tau))d\tau\right\}ds. \tag{3.18}$$

using (3.17), (3.3) and assumptions of the Theorem, we have

$$\|J_1\|\leq\int_{0}^{t}\|T(t-s)\|\left\|I-T\left(\hat{t}-t\right)\right\|\left\|f(s,u(s))+\int_{0}^{s} a(s,\tau)g(\tau,u(\tau))d\tau\right\|ds$$

$$\leq M\left\|I-T\left(\hat{t}-t\right)\right\|\left[\int_{0}^{t}\|f(s,u(s))-f(s,0)\|ds\right.$$

$$\left.+\int_{0}^{t}\int_{0}^{s}|a(s,\tau)|\,\|g(\tau,u(\tau))-g(\tau,0)\|d\tau ds\int_{0}^{t}\left\{\|f(s,0)\|+\int_{0}^{s}|a(s,\tau)|\|g(\tau,0)\|d\tau\right\}ds\right]$$

$$\leq M\left\|I-T\left(\hat{t}-t\right)\right\|\left[\int_{0}^{t} h_{i\alpha}(M_1)\|u(s)\|ds+\int_{0}^{t}\int_{0}^{s}\beta h_{2\alpha}(M_1)\|u(\tau)\|d\tau ds\right.$$

$$+\int_0^t h_{1\alpha}(M_1)ds+\int_0^t\int_0^s \beta h_{2\alpha}(M_1)d\tau ds\Bigg]$$

$$\leq M\left\|I-T\left(\hat{t}-t\right)\right\| t\left[\left(h_{1\alpha}(M_1)+\beta h_{2\alpha}(M_1)\alpha/2(M_1+1)\right)\right]$$

$$\leq M\left\|I-T\left(\hat{t}-t\right)\right\| \alpha\left[\left(h_{1\alpha}(M_1)+\beta h_{2\alpha}(M_1)\alpha\right)(M_1+1)\right]$$

$$= C\alpha\left\|I-T\left(\hat{t}-t\right)\right\| \tag{3.19}$$

where $C=M(M_1+1)\left(h_{1\alpha}(M_1)+\beta h_{2\alpha}(M_1)\alpha\right)$. From continuity of T and (3.19), we get

$$\|J_1\|\leq \varepsilon/2 \;\; for \left|\hat{t}-t\right|<\delta_1 \text{ for some } \delta_1 . \tag{3.20}$$

Similarly, from (3.18), (3.3) and assumptions of Theorem, we obtain

$$\|J_2\|\leq \int_t^{\hat{t}} Mh_{1\alpha}(M_1)\|u(s)\|ds+\int_t^{\hat{t}}\int_0^s M\beta h_{2\alpha}(M_1)\|u(\tau)\|d\tau ds$$

$$+\int_t^{\hat{t}} M\left\{h_{1\alpha}(M_1)+\int_0^s \beta h_{2\alpha}(M_1)d\tau\right\}ds$$

$$\leq M\left\{\left(\hat{t}-t\right)\left[h_{1\alpha}(M_1)+\beta h_{2\alpha}(M_1)\frac{\left(t+\hat{t}\right)}{2}(M_1+1)\right]\right\} \tag{3.21}$$

Again by continuity of T and (3.21), we observe that

$$\|J_2\| \le \varepsilon/2 \tag{3.22}$$

for $|\hat{t}-t|<\delta_2>0$ for some $\delta_2>0.$

From (3.16), (3.20) and (3.22), we obtain

$$\|v(t)-v(\hat{t})\| \le \varepsilon, \; for |\hat{t}-t| < \min\{\delta_1,\delta_2\} \tag{3.23}$$

and $0 \le t < \hat{t} < \alpha$, therefore

$\lim_{t\to\alpha} V$ exists and consequently $\lim_{t\to\alpha} u(t)$ exists. Now we wish to extend u to the interval $[0,\alpha+\varepsilon)$ for some $\varepsilon>0$. Define $u_j(t)=u(t)$, for $t\in[0,\alpha]$ and j=0,1,........ For $t>\alpha$, define

$$u_0(t)=T(t)u_0+\int_\alpha^t T(t-s)f\big(s,u(s)\big)ds+\int_0^t T(t-s)f(s,u(s))ds$$

$$+\int_\alpha^t T(t-s)\int_0^\alpha a(s,\tau)g(\tau,u(\tau))d\tau ds \tag{3.24}$$

and

$$u_{j+1}(t)=u_0(t)+\int_\alpha^t T(t-s)\int_\alpha^s a(s,\tau)g(\tau,u_j(\tau))d\tau ds+\int_\alpha^t T(t-s)f(s,u_j(s))ds. \tag{3.25}$$

Using the method similar to the proof of our Thorem 1, it can be shown that for some $t_0>\alpha$, the sequence $\{u_j(t)\}$ defined by (3.24)-(3.25), converges strongly to a function u uniformly on $t\in[0,t_0]$. Taking limit as $j\to\infty$ in (3.25) we get,

$$u(t)=u_0(t)+\int_d^t T(t-s)\int_\alpha^s a(s,\tau)g\big(\tau,u(\tau)\big)d\tau ds+\int_\alpha^t T(t-s)f\big(s,u(s)\big)ds. \tag{3.26}$$

Now, we shall prove that $u(t)$ given by (3.26) satisfies (1.1).

Applying Lemma 1 to (3.24) and (3.26), we obtain,

$$u_0'(t) = \frac{\partial}{\partial t} T(t)u_0 + T(t-\alpha) f(\alpha, u(a)) + \int_\alpha^t T(t-s) f'(s, u(s)) ds$$

$$+T(t-\alpha)\int_0^\alpha a(s,\tau) g(\tau, u(\tau)) d\tau + \int_\alpha^t T(t-s) \frac{\partial}{\partial s}\left(\int_0^\alpha a(s,\tau) g(\tau, u(\tau)) d\tau\right) ds$$

$$+T(t) f(0, u(0)) + \int_0^t T(t-s) f'(s, u(s)) ds \tag{3.27}$$

$$-Au_0(t) = -AT(t)u_0 - T(t-a) f(s, u(\alpha)) + f(t, u(t)) - \int_\alpha^t T(t-s) f'(s, u(s)) ds$$

$$-T(t-\alpha)\int_0^\alpha a(s,\tau) g(\tau, u(\tau)) dt + \int_0^\alpha a(s,\tau) g(\tau, u(\tau)) dt$$

$$-\int_\alpha^t T(t-s) \frac{\partial}{\partial s}\left(\int_0^\alpha a(s,\tau) g(\tau, u(\tau)) d\tau - T(t) f(0, u(0))\right)$$

$$-\int_0^t T(t-s) f'(s, u(s)) ds + f(t, u(t)), \tag{3.28}$$

and

$$u'(t) = u_0'(t) + T(t-\alpha) f(\alpha, u(\alpha)) + \int_\alpha^t T(t-s) f'(s, u(s)) ds$$

$$+\int_\alpha^t T(t-\alpha) \frac{\partial}{\partial s}\left(\int_0^s a(s,\tau) g(\tau, u(\tau)) d\tau\right) ds, \tag{3.29}$$

$$-Au(t) = -Au_0(t) - T(t-\alpha) f(\alpha, u(\alpha)) + f(t, u(t)) - \int_\alpha^t T(t-s) f'(s, u(s)) ds$$

$$+\int_{\alpha}^{t} a(t,s)g(s,u(s))ds \; -\int_{\alpha}^{t} T(t-s)\frac{\partial}{\partial s}(\int_{\alpha}^{s} a(s,\tau)g(\tau,u(\tau))d\tau)ds$$

Using (3.27) - (3.30) and $\frac{\partial}{\partial t}T(t)u_0 = AT(t)u_0$, (a property of semi-group), we get,

$$u'(t) - Au(t) = \int_{0}^{\alpha} a(s,\tau)g(\tau,u(\tau))d\tau + f(t,u(t)) + \int_{0}^{t} a(t,s)g(s,u(s))ds. \tag{3.31}$$

This can also be written as

$$u(t) + Au(t) = f(t,u(t)) + \int_{0}^{t} a(t,s)g(s,u(s))ds,$$

where $-A$ is an infinitesimal generator of c_0-semigroup which showes that u satisfies (2.3) on $[0,t_0]$. The proof of the theorem 3 is complete.

PROOFS OF THEOREMS 4 AND 5

The mild solutions of (1.1 and (2.1) are given by (2.3) and

$$z(t) = T(t)z_0 + \int_{0}^{t} T(t-s)F(s,z(s))ds + \int_{0}^{t} T(t-s)\int_{0}^{s} a(s,\tau)G(\tau,z(\tau))d\tau ds, \tag{4.1}$$

respectively. From (2.3), (4.1), assumptions (2.4) - (2.6) and hypotheses (H_1)-(H_2), we get

$$\|u(t)-z(t)\| \le \|T(t)\|\|u_0 - z_0\| + \int_{0}^{t}\|T(t-s)\|\,\|f(s,u(s)) - f(s,z(s))\|ds$$

$$+\int_{0}^{t}\|T(t-s)\|\,\|f(s,z(s)) - F(s,z(s))\|ds$$

$$+\int_{0}^{t}\|T(t-s)\| \int_{0}^{s}|a(s,\tau)|\|g(\tau,u(\tau)) - g(\tau,z(\tau))\|d\tau ds$$

$$+\int_0^t \|T(t-s)\| \int_0^s |a(s,\tau)| \|g(\tau,z(\tau)) - G(\tau,z(\tau))\| d\tau ds$$

$$\leq M\|u_0 - z_0\| + \int_0^t Mh_1(k)\|u(s)-z(s)\|ds$$

$$+\int_0^t M \in ds + \int_0^t M \int_0^s \beta h_2(k)\|u(\tau)-z(\tau)\|d\tau ds + \int_0^t M \int_0^s \beta \in d\tau ds.$$

$$\leq M \in \left(1+t+\beta t^2/2!\right) + \int_0^t Mh_1(k)\|u(s)-z(s)\|ds$$

$$+\int_0^t Mh_1(k) \int_0^s \beta \frac{h_2(k)}{h(k)} \|u(\tau)-z(\tau)\|d\tau ds$$

$$= h(t) + \int_0^t Mh_1(k)\|u(s)-z(s)\|ds + \int_0^t Mh_1(k) \int_0^s \frac{\beta h_2(k)}{h_1(k)} \|u(\tau)-z(\tau)\|d\tau ds \tag{4.2}$$

where $\|T(.)\| \leq M$, $\|u(.)\|$, $\|z(.)\| \leq k$ and $h(t) = M \in \left(1+t+\beta t^2/2!\right)$. It can be very easily seen that the function $h(t)$ is positive, monotonic, nondecreasing and continuous. Using lemma 2, we get

$$\|u(t)-z(t)\| \leq h(t)\left\{1 + \int_0^t Mh_1(k)\exp\left(\int_0^s Mh_1(k) + \frac{\beta h_2(k)}{h_1(k)}\right)d\tau)ds\right\}$$

$$= h(t)\left\{1 - \frac{M(h_1(k))^2}{.\left[M(h_1(k))^2 + \beta h_2(k)\right]}\left[1 - \exp\left(\left\{\left(M(h_1(k))^2 + \beta h_2(k)\right)/h_1(k)\right\}t\right)\right]\right\}. \tag{4.3}$$

On the compact set right hand side of (4.3) is bounded, so the solution of the initial value problem (1.1) depends continuously on the functions involved in (1.1) and initial value. Moreover, if $\varepsilon \to 0$, then $\|u(t)-v(t)\| \to 0$. Hence the proof of theorem 4 is complete.

The mild salutions of (1.1) and (2.2) are given respectively by (2.3) and

$$v(t) = T(t)v_0 + \int_0^t T(t-s) f(s, v(s))ds + \int_0^t T(t-s)\int_0^s a(s,\tau) g(\tau, v(\tau))d\tau ds$$

$$+\int_0^t T(t-s) H(s, v(s))ds, \tag{4.4}$$

From (2.3), (4.4) assumption (2.7) of the Theorem 5 hypotheses (H_1)-(H_2), (H_4) and boundedness of solution u of (1.1) i.e. $\|u(t)\| \leq N$, and $\|T(.)\| \leq M$ on $[0,\alpha]$ we obtain,

$$\|u(t) - v(t)\| \leq \|T(t)\| \|u_0 - v_0\| + \int_0^t \|T(t-s)\| f(s, u(s)) - f(s, v(s))\| ds$$

$$+\int_0^t \|T(t-s)\| \int_0^s |a(s,\tau)| \|g(\tau, u(\tau)) - g(\tau, v(\tau))\| d\tau ds$$

$$+\int_0^t \|T(t-s)\| \|H(s, v(s))\| ds$$

$$\leq M\|u_0 - v_0\| + \int_0^t Mh_1(k)\|u(s) - v(s)\| ds + \int_0^t M \int_0^s \beta h_2(k) \|u(\tau) - v(\tau)\| d\tau ds.$$

$$+\int_0^t ML\left[\|u(s) - v(s)\| + \|u(s)\|\right] ds$$

$$\leq M \in (1 + LNt) + \int_0^t M(h_1(k) + L\varepsilon)\|u(s) - v(s)\| ds$$

$$+\int_0^t M(h_1(k) + L\varepsilon)\int_0^s \frac{\beta h_2(k)}{(h_1(k) + L\varepsilon)} \|u(\tau) - v(\tau)\| d\tau ds \tag{4.5}$$

It can be seen that $M \in (1 + LNt)$, is positive monotonic, nondecreasing and continuous, therefore using Lemma 3, we obtain

$$\|u(t) - v(t)\| \leq M \in (1 + LNt)$$

$$\left\{1 + \int_0^t M(h_1(k) + L\varepsilon)\exp(\int_0^s (M(h_1(k) + L\varepsilon) + \frac{\beta h_2(k)}{(h_1(k) + L\varepsilon)})d\tau)ds\}\right\}$$

$$= M(1 + LNt) \in \left\{1 - \frac{(Mh_1(k) + L\varepsilon)}{\left[(h_1(k) + L\varepsilon) + \frac{\beta h_2(k)}{(h_1(k) + L\varepsilon)}\right]}(1 - \exp[((h_1(k) + L\varepsilon) + \frac{\beta h_2(k)}{(h_1(k) + L\varepsilon)})t])\}\right\}$$

(4.6)

Note that on a compact set, the right hand side of (4.6) is bounded and since ε is an arbitrary, therefore as $\varepsilon \to 0$, we get $\|u(t) - v(t)\| \to 0$, which gives an equivalence between the solutions of (1.1) and (2.2). Theorem 5 is complete.

EXAMPLES

$$w_t(x,t) = (\phi(x)w_x(x,t))_x$$

$$= P(t, w(x,t)) + \int_0^t a(t,s)Q(s, w(x,s))ds, \qquad 0 < x < 1, t > 0, \tag{5.1}$$

with the given initial and boundary conditions

$$\begin{aligned} w(0,t) &= w(1,t) = 0 && 0 \leq t \leq \alpha \\ w(x,0) &= w_0(x) && 0 \leq x \leq 1 \end{aligned} \tag{5.2}$$

Where the functions $a : [0,\alpha] \times [0,\alpha] \to R,$ is a Hölder continuous with exponent and the functions $P, Q : [0,\alpha] \times R \to R$ and $\phi : R \to R$ are continuous and satisfy

the following conditions;

$$\left.\begin{aligned} &\text{(i)} \quad |P(t,z_1)| \le m_1(t^*), \\ &\text{(ii)} \quad |Q(t,z_1)| \le m_2(t^*), \\ &\text{(iii)} \quad |P(t,z_1) - P(t,z_2)| \le m_1(t^*)|z_1 - z_2|, \\ &\text{(iv)} \quad |Q(t,z_1) - Q(t,z_2)| \le m_2(t^*)|z_1 - z_2|, \end{aligned}\right] \tag{5.3}$$

where $m_i : R_+ \to R$, $i = 1,2$ are positive nondecreasing functions $z_i \in R$, $i = 1,2$; $|z_1|, |z_2| \le t^*$ and $t \in [0,\alpha]$.

We first reduce the problem (5.1) - (5.2) to the form (1.1) by making suitable choices of *A, f, a* and *g* and then illustrate the hypotheses of our Theorems 1 and 5, established in section 3.

Let $X = L^{'}[0,1], p > 2.$ We define an operator $A : X \to X$, by

$(Au(t))(x) = -(\phi(x) w_n(x,t))_x$ with

$D(A) = \{x \in X; (\phi(.) w_x(,))_x \in X | w(0,t) = w(1,t) = 0\},$

and the function $f : R_+ \times X \to X \, and \, g : R_+ \times X \to X$ are defined as follows

$$\left.\begin{aligned} f(t,z)(x) &= P(t,z(x)), \\ g(t,z)(x) &= Q(t,z(x)), \end{aligned}\right\} \tag{5.4}$$

for $(t,z) \in R_+ \times X$ and $0 < x < 1$.

From the above choices of the functions and generator A, the equation (5.1)-(5.2) can be formulated as

$$\left.\begin{aligned} u^{'}(t) + Au(t) &= f(t,u(t)) + \int_0^t a(t,s) g(s,u(s)) ds \\ u(0) &= u_0 \end{aligned}\right\} \tag{5.5}$$

Since all the Hypothesis of Theorem 1 are satisfied therefore Theorem 1 can be applied to guarantee the existence and uniqueness of the solution of (5.1)-(5.2).

Example 2. In this example, we shall obtain a behavioural relationship between the solutions of partial integrodifferential equations (5.1)-(5.2) and the perturbed partial integrodifferential equations of the form,

$$z_t(x,t)-(\phi(x)z_x(x,t))_x = P(t,z(x,t))+\int_0^t a(t,s)Q(s,z(x,s))ds$$

$$+E(t,z(x,t)) \quad 0<x<1, \ t\geq 0. \tag{5.6}$$

With the given initial and boundary conditions

$$\left.\begin{array}{l} z(0,t) = z(1,0) = 0 \\ z(x,0) = z_0(x), \quad 0 < x < 1, \ t \geq 0, \end{array}\right\} \tag{5.7}$$

where the functions a, ϕ ,P and Q are as defined in the example 1, the functions P and Q satisfy the conditions of (5.3) and E: $[0,\alpha]\times R \to R$, satisfies the following condition :

$$|E(t,z)| \leq L \in |z|, \tag{5.8}$$

where L is a nonnegative constant, $\varepsilon > 0$ is an arbitray, $z \in R$ and $t \in [0,\alpha]$.

Let $X=$ $L^p[0, 1]$, $p>2$ We define, an operator $A\colon X \to X$ by

$$(Av(t))_x = -(\phi(x)z_x(x,t) \text{ with}$$

$$D(A) = x \in X, \ (\phi(x)Z_x(,)_x \in X) \quad Z(0,t) = Z(1,t),$$

the functions g, f: $R_+ \times X \to Xh_y$ (5.4) and

$H: R_+ \times X \to X$ as follows

$$H(t,z)(x) = E(t,z(x) \tag{5.10}$$

for $(t, z) \in R_+ \times X$ and $0 < x < 1$. With these definitions of the functions *f, g* and *H* and operator A, the equation (5.6) - (5.7) can be formulated abstractly as

$$\left.\begin{aligned} & v'(t) + Av(t) = f(t, v(t)) + \int_0^t a(t,s) g(s, v(s)) ds + H(t, v(t)) \\ & v(0) = v_0 \end{aligned}\right\} \qquad (5.11)$$

Since all the conditions of Theorem 5 are fulfilled, hence we get the equivalence relation between the solutions of (5.1)-(5.2) and (5.6) - (5.7).

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NOISE ELIMINATION IN JOINT TRANSFORM CORRELATOR BY DIFFERENCING

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(Received 25.01.2000)

Keywords : Pattern Recognition, Differencing, Optical correlation.

INTRODUCTION

Optical correlation has been used for the detection and identification of objects in a particular image area. The presence or absence is indicated by the correlation peaks appearing in correlation plane. This method requires the development of a matched filter which involves holographic techniques and an involved process. It is also difficult to generate rapidly changing matched filters specially when results are required in quick succession. The joint transform correlation (JTC) technique does not require the development of a matched filter for obtaining optical correlation. This technique has great potential when fast results are required to be analysed. The JTC is also a non-real time phenomenon. Fourier transform (F, T) of the object and reference is obtained, which is again illuminated and FT of the FT (both object and image) is obtained. The FT thus obtained has the required correlation peaks which can be analysed.

The output of a JTC correlator is highly noisy. The DC portion itself occupies lot of space and the peaks are submerged in the noise. This paper suggests a simple method for eliminating the noise in a JTC. The scheme is based upon the binarising the image and taking the difference of two JTC output correlation peaks.

JOINT TRANSFORM CORRELATOR

The schematic representation of joint transform correlator is shown in figure 1 & 2 . The input functions g and h which are to be correlated are placed in the input plane side by side as shown in figure 1.The amplitude transmittance of input plane in one dimension is given by

$$U_1(x_1, y_1) = g\,(x_1, y_1 + b) + h\,(x_1, y_1 - b)$$

where $2b$ is the centre to centre spacing of the two functions and the physical limitation of them is to be "b". The lens L_1 forms the FT of U_1 at FT Plane P_2.

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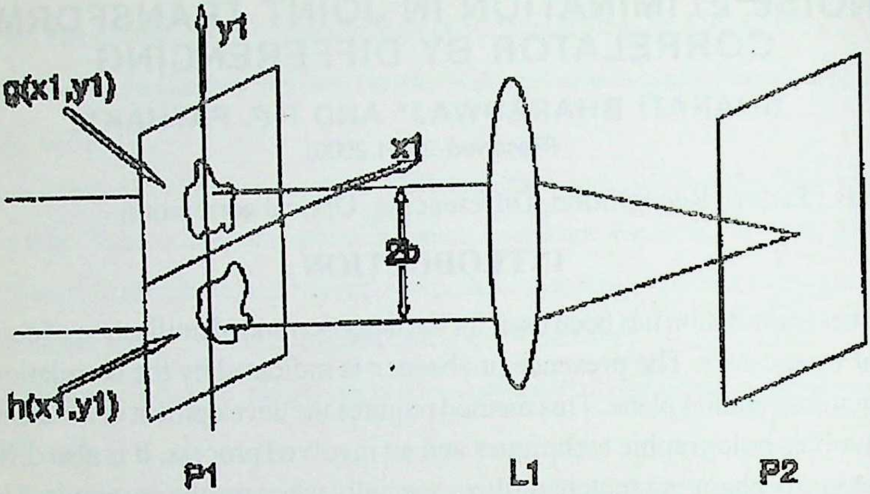

Fig. 1 : Fourier Transform of Function g & h.

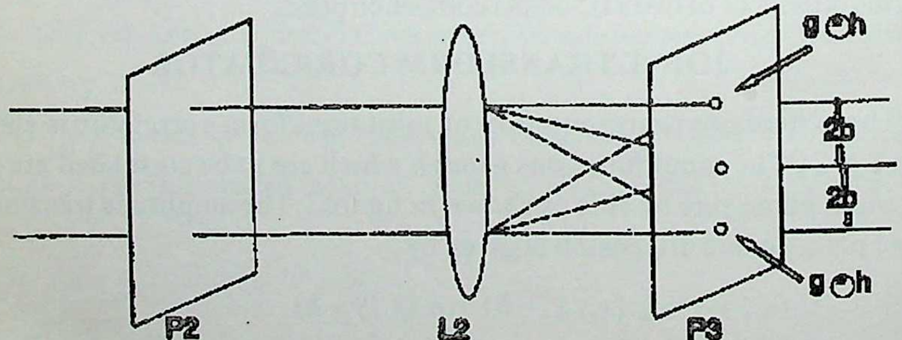

Fig. 2 : Joint Transform Correlation

$$U_2(x_2, y_2) = F[(u_1(x_1, y_1)]$$
$$= F[g(x_1, y_1+b)+h(x_1, y_1-b)]$$
$$= G \exp[j2\pi vb]+H \exp[-j2\pi vb]$$

The transmittance of P_2, assuming it to be proportional to the light intensity, is given by

$$t(x_2, y_2) = |U_2(x_2, y_2)|^2$$
$$= |G(u,v)\exp(+j2\pi vb) + H(u,v)\exp(-j2\pi vb)|^2$$
$$= |G|^2 + |H|^2 + GH^* \exp(+j4\pi vb)$$
$$+ G^* H \exp(-j4\pi vb)$$

where $u = \dfrac{x}{\lambda f}$ and $v = \dfrac{y}{\lambda f}$

When this function $t(x_2, y_2)$ is illuminated as shown in fig. 2. The FT of $t(x_2, y_2)$ is formed at plane P_3 by lens L_2.

At plane P_3 the light amplitude distribution is the FT of $t(x_2, y_2)$

$$U(x_3, y_3) = g \cdot g + h \cdot h + g \cdot h\delta(x_3, y_3 + 2b) + h \cdot g\delta(x_3, y_3 - 2b)$$

where $\cdot$ is correlation.

It is thus clear that $U(x_3, y_3)$ contains the designed correlation spots centred at $(x_3, y_3 \pm 2b)$, equal focal lengths of L_1 and L_2 are assumed for simplicity.

DIFFERENCE SCHEME FOR NOISE ELIMINATION

The first two terms in last equation are DC and the 3rd and 4th terms are related to the cross correlation peaks.

The noise elimination is based upon obtaining a diference of two JTC correlation output peaks. The JTC output of an input having target and reference is shown in fig. 3 a & 3b. The target and reference images are rotated by an angle θ, in this experiment say by 45^0. The correlation peaks are observed again. The rotated target and reference alongwith the observed peaks in JTC output are shown in fig. 4 a and 4b. The output of fig 1 and 2 are digitally scanned and registered. A difference image is created by plotting $|A_{ij} - B_{ij}|$ where A_{ij} and B_{ij} are the intensites of pixels in the

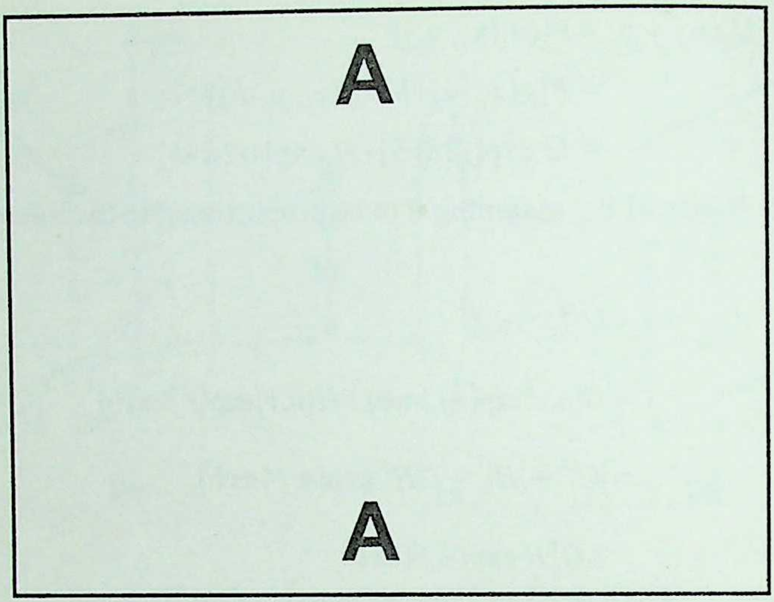

Fig. 3 (a) - Target and Reference

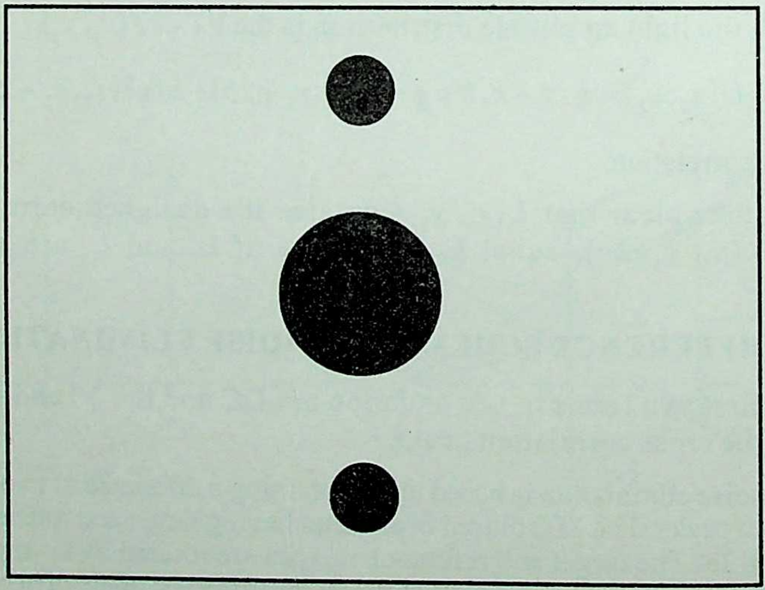

Fig. 3 (b) - Correlation Peaks

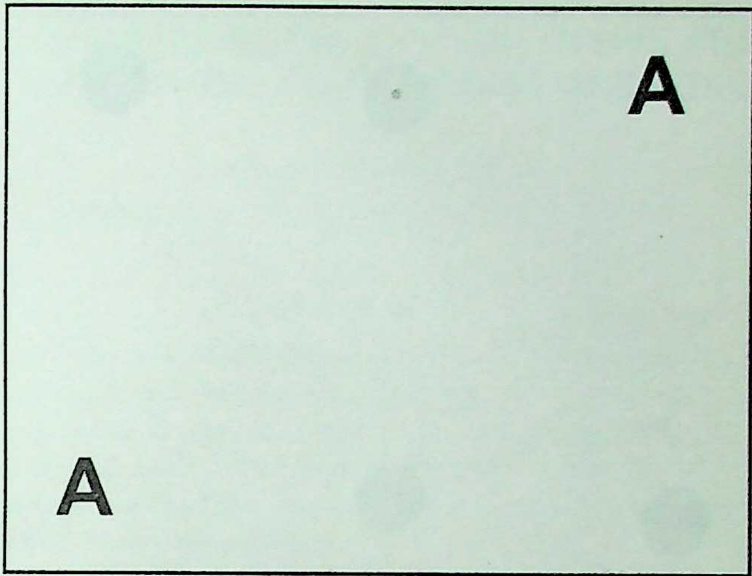

Fig. 4 (a) - Target and Reference Rotated by an angle 45^0

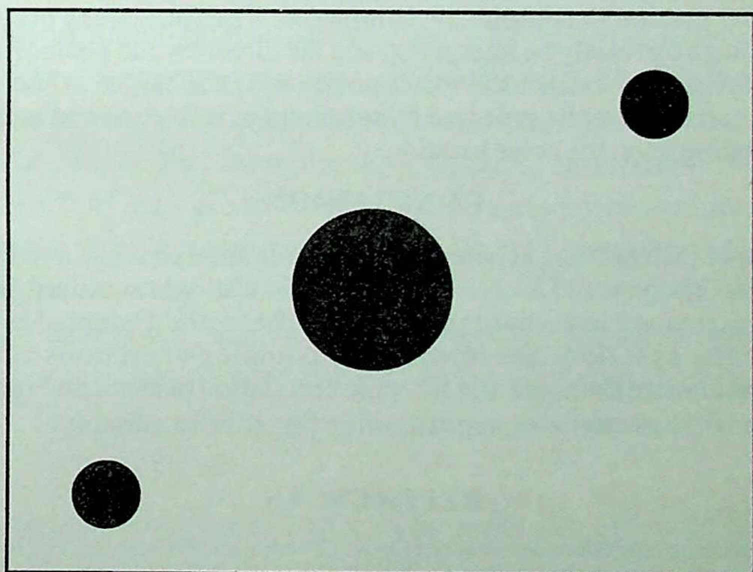

Fig. 4 (b) - Correlation Peaks

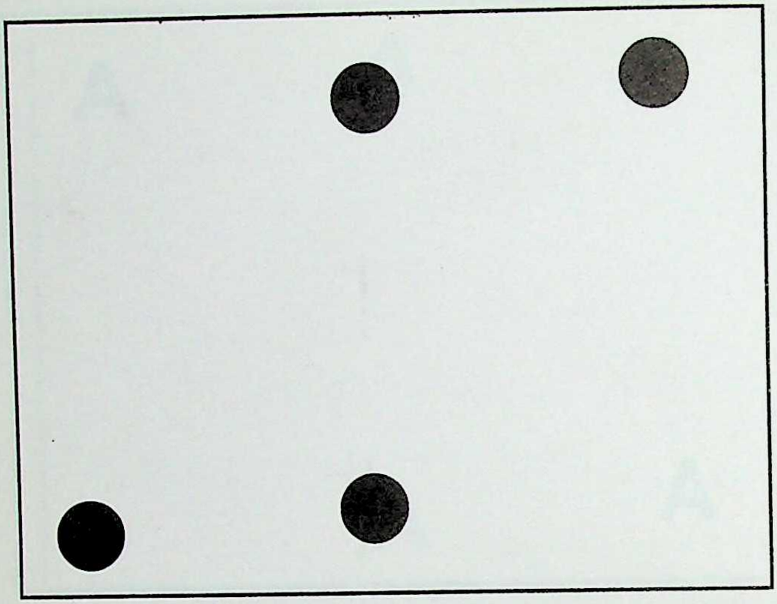

Fig. 5 - Difference of 3(b) and 4(b) $|Aij - Bij|$

two JTC outputs in the i^{th} row and j^{th} columns. The difference image created by $|Aij - Bij|$ gives all the four correlation peaks of two JTC output, while the noise including DC portion is completely eliminated. The four peaks observed in the difference image can easily be interpreted and the direction and position of the peaks can be easily used to extract the exact position of the target in the image. The experiment can however be repeated by rotating the reference and targets at other angles depending upon the noise location.

CONCLUSION

A simple differencing scheme for the elimination of noise and DC in JTC is presented in this paper. The scheme is very useful when output peaks in the correlation planes are immensed in the noise. The method adopted in the scheme is a simple one as it does not involve any complex operations and is able to completely eliminate the noise and filter the correlation peaks with high efficiency. The scheme will have definite application in finger print matching.

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Journal of Natural & Physical Sciences Vol. 14(1-2) (2000) 37-48

A STOCHASTIC MODEL OF A DIFFUSIVE PREY-PREDATOR SYSTEM : FLUCTUATION AND STABILITY

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(Received 15.01.1999 and after revision 12.03.2000)

ABSTRACT

A stochastic analysis of stability of a diffusive Prey-Predator system under the influence of randomly fluctuating environment has been investigated. The analysis had led to the scenarios of stability or instability different from those based on deterministic model of the system. The numerical values of variance and auto-correlation function of the population have been calculated to present a graphical picture of fluctuating pattern.

Mathematics Subject Classification : 92A17.

Key-Words : Prey-predator system, Diffusion, Fluctuating environment, Spectral density, Fluctuation and stability.

INTRODUCTION

The study of spatial interaction between two or more species has recently received considerable attention in ecological literature [3]. The stochasticity plays a vital role in such situation and is of particular importance in the context of marine population [2]. For example, the stochastic simulation model due to Duboise Manfort [4], of a generalized diffusive Lotka-Volterra predator-prey system describing the interaction of phytoplankton (prey) and herbivorous zooplankton (predator) showed that it led to the spontaneous emergence of prey strong spectral heterogeneity similar to those observed in the sea. The stochasticity is, therefore, of special significance in depicting the real picture of a heterogeneous diffusive system which is under the constant influence of randomly fluctuating environment.

The object of the present paper is to study the effect of a randomly fluctuating environment on the behaviour, particularly on the stability of a diffusive

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prey-predator system. The effect of diffusion in absence of stochastic perturbation may sometimes leads to instability called diffusive instability of an otherwise stable system [2]. The scenario of stability may further change with the addition of stochasticity in the basic model equation of the system. The present paper aims to investigate such a case for the diffusive prey-predator system under the influence of randomly fluctuating environment.

The stability analysis for the stochastic system is based on the concept of second order moment [7, 8]. Finally the auto-correlation function of the prey-predator population have been calculated numerically and presented graphically for a better understanding of the pattern (both spatial and temporal) of fluctuation.

PREDATOR-PREY DIFFUSIVE SYSTEM : DETERMINISTIC MODEL

We consider a one-dimensional diffusive prey-predator model. Let the local interaction between populations of the prey (N_1) and the predator (N_2) be characterized by the system of two partial differential equations [1].

$$\frac{\partial N_1}{\partial t} = N_1\left[\frac{\varepsilon N_1}{1+N_1} - N_2\right] + D_1\frac{\partial^2 N_1}{\partial x^2}$$

$$\frac{\partial N_2}{\partial t} = N_2\left[N_1 - \gamma N_2\right] + D_2\frac{\partial^2 N_2}{\partial x^2} \tag{1}$$

where D_i (>0) are diffusion coefficients, $\varepsilon > 0$, $\gamma > 0$. For simplicity, here we have considered only one-dimensional process. In reality ecological diffusion occurs in two or three dimensional space resulting in patchy distribution of population. For isotropic behavior of the patchy distribution, the one-dimensional diffusive analysis will exhibit the same pattern as that of the higher dimensional analysis. A well-documented example is the case of Phytoplanktons (prey) and zooplanktons (predator)-microscopic aquatic organisms, often found in even distribution at or close to the surface of water [2]. A well-known one dimensional analysis of such a diffusive prey-predator model which exhibits patchiness is due to Mimura and Murray [11]. The non-trivial non-diffusive equilibrium has coordinates $N_1^* = \varepsilon\gamma - 1$, $N_2^* = \frac{\varepsilon\gamma - 1}{\gamma}$. It can be shown that the local equilibrium

(N_1^*, N_2^*) is stable if [1].

$$\varepsilon\gamma^2 > 1, \text{ and } \varepsilon\gamma > 1 \tag{2}$$

We now perform stability analysis of uniform state with a much larger class of perturbations which depend both on space as well as time :

$$N_1(x,t) = N_1^* + n_1(x,t) \tag{3}$$

$$N_2(x,t) = N_2^* + n_2(x,t)$$

The linearized perturbation equations are as follows :

$$\frac{\partial n_1}{\partial t} = A_{11}n_1 + A_{12}n_2 + D_1\frac{\partial^2 n_1}{\partial x^2}$$

$$\frac{\partial n_1}{\partial t} = A_{21}n_1 + A_{22}n_2 + D_2\frac{\partial^2 n_2}{\partial x^2} \tag{4}$$

Where $A_{11} = \dfrac{\varepsilon\gamma - 1}{\varepsilon\gamma^2}$, $A_{12} = A_{22} = -(\varepsilon\gamma - 1)$, $A_{21} = \dfrac{\varepsilon\gamma - 1}{\gamma}$ (5)

We assume the solution of (4) to be of the form

$$n_1(x,t) = \hat{x}(t)\cos(px), n_2(x,t) = \hat{y}(t)\cos(px) \tag{6}$$

We substitute (6) in (4) in order to verify that (6) is a possible form of solution and we observe that the equations (4) are indeed satisfied [after the cancellation of the common factor cos (px)] if

$$\frac{d\hat{x}}{dt} = (A_{11} - D_1p^2)\hat{x} + A_{12}.\hat{y}$$

$$\frac{d\hat{y}}{dt} = A_{21}.\hat{x} + (A_{22} - D_2p^2)\hat{y} \tag{7}$$

which is a system of linear ordinary differential equations with constant coefficients. The exponential solutions of (7) decay to zero iff

$$\left(A_{11}-D_1p^2\right)\left(A_{22}-D_2p^2\right)-A_{12}A_{21}>0$$

and

$$\left(A_{22}-D_1p^2\right)\left(A_{11}-D_2p^2\right)>0 \tag{8}$$

implying

$$D_1D_2p^4+\frac{\varepsilon\gamma-1}{\varepsilon\gamma^2}\left(\varepsilon\gamma^2D_1-D_2\right)p^2+\frac{(\varepsilon\gamma-1)^3}{\gamma}>0 \tag{9}$$

Since $\varepsilon\gamma^2>1$ and $\varepsilon\gamma>1$ (9) will certainly hold if $\quad D_2<\varepsilon\gamma^2D_1 \quad$ (10)

Thus for spatially non-homogeneous case to be stable, condition (10) must hold. The following three cases arise according to the values of D_1 and D_2:

(i) If both D_1 and D_2 tends to zero, ie, in the case of no diffusion, the criterion (9) of stability implies either $\varepsilon\gamma>1$ and $\gamma>0$ or $\varepsilon\gamma<1$ and $\gamma<0$. The second condition is impossible according to our condition laid down earlier. So, for the stability of the non-diffusive homogeneous system, the criterion of stability becomes $\varepsilon\gamma>1$ and $\gamma>0$, which is one of the criterion for stability of the non-diffusive or homogeneous system.

(ii) If $D_1\longrightarrow 0$, ie, if the mobility of the prey is vanishingly small, then the inequality (10) cannot be satisfied for finite values of D_2, as such the heterogeneous steady state of the system is not stable. This corresponds to the case of diffusive instability of the system [5].

(iii) If $D_2\longrightarrow 0$, ie., if the mobility of the predator is vanishing small, then the inequality (10) can be satisfied for finite values of D_1. As such the corresponding heterogeneous steady state is stable.

In the next section, we shall study the validity or invalidity of the above criteria in the case of inhomogeneous diffusive system under the influence of random fluctuating environment.

DIFFUSIVE SYSTEM : STOCHASTIC MODEL

Now taking into account the one-dimensional diffusion of fluctuating environment, we modify the system (1) to the form of the partial stochastic differential equation (in the stratonovich sense) [6] :

$$\frac{\partial N_1}{\partial t} = N_1\left[\frac{\varepsilon N_1}{1+N_1} - N_2\right] + D_1\frac{\partial^2 N_1}{\partial x^2} \equiv G_1(N_1, N_2, \gamma_1)$$

$$\frac{\partial N_2}{\partial t} = N_2[N_1 - \gamma_1 N_2] + D_2\frac{\partial^2 N_2}{\partial x^2} \equiv G_2(N_1, N_2, \gamma_1) \qquad (11)$$

where $\gamma_1 = \gamma + \phi(x,t), \gamma$ is the deterministic part, $\phi(x,t)$ is an environmental parameter and we assume that it is a white noise of unit spectral density. The boundary conditions are selected in such a way that when $\phi(x,t)$ is set everywhere equal to zero, the system has a stable and spatially uniform stationary state $\left[\varepsilon\gamma - 1,\ \frac{\varepsilon\gamma - 1}{\gamma}\right]$.

If the spectral densities of the fluctuating prey and predator population be S_1 and S_2 respectively and that of the driving environment be S_ϕ, then it can be shown [9]

$$S_m(k,\omega) = |T_m(k,\omega)|^2 S_\phi(k,\omega) \qquad (m=1,2) \qquad (12)$$

where $T_m(k,\omega)$ is the transfer function. Since the driving fluctuation ϕ has been assumed to be a white noise of unit spectral density, ie., $S_\phi = 1$.

$$\text{Thus } S_m(k,\omega) = |T_m(k,\omega)|^2 \quad (m = 1,2) \qquad (13)$$

A bit of calculation leads to the transfer function T_m (k,ω) [9]

$$\left.\begin{aligned} |T_1(\omega,k)|^2 &= \frac{\alpha_2^2 A_{12}^2}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega\beta)^2} \\ |T_2(\omega,k)|^2 &= \frac{\left\{\left(D_1 k^2 - A_{11}\right)^2 + \omega^2\right\}\alpha_2^2}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega\beta)^2} \end{aligned}\right] \qquad (14)$$

where

$$\omega_0^2(k) = M - (D_1 A_{22} + D_2 A_{11})k^2 + D_1 D_2 k^4;\; M = A_{11}A_{22} - A_{12}A_{21} \;;\; c = -A_{11} - A_{22};$$

$$\beta(k) = c + (D_1 + D_2)k^2 ; \alpha_m = \left(\frac{\partial G_m}{\partial \gamma_1}\right) N_1^*, N_2^* \;;\; \alpha_1 = 0,\, \alpha_2 = \frac{(\varepsilon\gamma - 1)^2}{\gamma^2} \tag{15}$$

The intensity (variance) of the fluctuations are given by [9]

$$\sigma_m^2 = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|T_m(k,\omega)\right|^2 dk\, d\omega \quad (m=1,2) \tag{16}$$

where σ_1^2, σ_2^2 represent the intensity of the fluctuations of prey and predator respectively.

Therefore,

$$\sigma_1^2 = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|T_1(k,\omega)\right|^2 dk\, d\omega = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\left|\frac{\alpha_2^2 A_{12}^2}{(\omega_0^2 - \omega^2)^2 + (\omega\beta)^2}\right|^2 d\omega\right\} dk$$

After some calculations [10], $\sigma_1^2 = \frac{\sigma_2^2 A_{12}^2}{4\pi}\int_{-\infty}^{\infty}\frac{dk}{\beta\omega_0^2} = \frac{P(D_1, D_2)}{Q(D_1, D_2)}$ (17)

where $P(D_1, D_2) = \sigma_2^2 A_{12}^2\left[\sqrt{cD_1 D_2} + \left\{(D_1 + D_2)\left(2\sqrt{MD_1 D_2} - D_1 A_{22} - D_2 A_{11}\right)\right\}^{1/2}\right]$

$$Q(D_1, D_2) = 4\sqrt{MC}(2\sqrt{MD_1 D_2} - D_1 A_{22} - D_2 A_{11})^{1/2}\Big[(D_1 + D_2)\sqrt{M} + C\sqrt{D_1 D_2}$$
$$\left(2\sqrt{MD_1 D_2} - D_1 A_{22} - D_2 A_{11}\right)^{1/2}\sqrt{C(D_1 + D_2)}$$

Similarly,

$$\sigma_2^2 = \frac{\sigma_2^2}{4\pi}\int_{-\infty}^{\infty}\frac{D^2 k^4 + \omega_0^2}{\beta\omega_0^2} dk = \frac{\sigma_2^2}{4}\left[\frac{P'(D_1, D_2)}{Q'(D_1, D_2)} + \frac{1}{\sqrt{C(D_1 + D_2)}}\right] \tag{19}$$

where

$$P'(D_1, D_2) = \left(2\sqrt{MD_1D_2} - D_1A_{22} - D_2A_{11}\right)^{1/2}\left\{(D_1 + D_2)^2 A_{11}\sqrt{D_1} + D_1^{3/2}C\sqrt{M}\right\}$$

$$+A_{11}\left(\sqrt{D_2} - \sqrt{MD_1}\right)^2\sqrt{CD_1(D_1 + D_2)} \tag{20}$$

$$Q(D_1, D_2) = \left\{MCD_2(D_1 + D_2)\left(2\sqrt{MD_1D_2} - D_1A_{22} - D_2A_{11}\right)\right\}^{1/2}$$

$$\left[(D_1 + D_2)\sqrt{M} + C\sqrt{D_1D_2} + \left(2\sqrt{MD_1D_2} - D_1A_{22} - D_2A_{11}\right)^{1/3}\sqrt{C(D_1 + D_2)}\right]$$

Let us now consider the following cases :

Case I

We take $\mathbf{D}_i = \mathbf{S}_i\mathbf{D}$, then we have

$$\sigma_1^2 = \frac{P(s_1, s_2)}{\sqrt{D}Q(s_1, s_2)} \tag{21}$$

$$\sigma_2^2 = \frac{\sigma_2^2}{4\sqrt{D}}\left[\frac{P'(s_{1,}, s_2)}{Q'(s_{1,}, s_2)} + \frac{1}{\sqrt{C(s_{1,}, s_2)}}\right] \tag{22}$$

From (21) and (22) we see that as $D \to 0^+$, both the population variances σ_1^2 and σ_2^2 tends to infinity. This implies that both prey and predator population have fluctuation so severe that it leads to complete shattering of the system implying extinction of the population. This is a case of statistical instability and is completely different from its deterministic counterpart.

Case II

If $D_1 \to 0^+$, keeping D_2 finite, then

$$\sigma_1^2 = \frac{\alpha_2^2 A_{12}^2}{4\sqrt{MCD_2}\left\{\sqrt{M} + \sqrt{-CA_{11}}\right\}} \tag{23}$$

$$\sigma_2^2 = \frac{\sigma_2^2}{4\sqrt{CD_2}}\left[\frac{A_{11}}{\left\{\sqrt{M}+\sqrt{-CA_{11}}\right\}\sqrt{M}}+1\right] \tag{24}$$

From (23) and (24) we observe that the fluctuations σ_1^2, σ_2^2 remain finite for $D_1 \to 0^+$ and finite values of D_2 (not very small). This leads to the conclusion that in a fluctuating environment the preypredator system will be stable in absence of spreading of preys and under the finite (not very small) diffusive velocity of predators. This is in contradiction with the deterministic model discussed in section 2.

Case III.

In this case we take $D_2 \to 0^+$ keeping D_1 finite but not very small, then

$$\sigma_1^2 = \frac{\alpha_2^2 A_{12}^2}{4\sqrt{MCD_1}\left\{M+\sqrt{-CA_{11}}\right\}} \tag{25}$$

$$\sigma_2^2 \to \infty \tag{26}$$

From (25) and (26) we conclude that the prey-predator system will be unstable in the absence of spreading of predators and under the finite (not very small) diffusive velocity of preys in a fluctuating environment. This is also in contradiction with the deterministic model of the diffusive system.

All the result obtained above shows that a randomly fluctuating environment plays a vital role in the existence or extinction of a diffusive prey-predator ecosystem. All these results and conclusions are based on the model system considered. Their practical validity, however, remains to be tested.

AUTO-CORRELATION FUNCTION : NUMERICAL SIMULATION

The variances σ_1^2 and σ_2^2 given by (17) and (19) determine the fluctuations of the prey and predator populations respectively. To determine the pattern (both spatial and temporal) of fluctuation, we need to calculate the auto-correlation function of the population [9]

$$\rho_m^2(\xi,\tau) = \frac{1}{(2\pi\sigma_m)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|T_m(k,w)|^2 \exp\{i(k\xi+\omega t)\}dkd\omega \quad (m=1,2) \tag{27}$$

Putting the explicit form of $|T_m(k,\omega)|$ from equation (14) and integrating with respect to ω we have

$$\rho_m^2 = \int_{-\infty}^{\infty} W_m \exp\left(-\frac{1}{2}\beta|\tau|\right)\cos(\omega_1\tau)\cos(k\xi)\, dk \qquad (m=1,2) \tag{28}$$

where

$$\omega_1^2 = \sqrt{\omega_0^2 - \frac{\beta^2}{4}}\,, \quad W_m = \frac{1}{4\pi\sigma_m^2}\left[\frac{(\alpha_n A_{mn} - \alpha_m A_{nn} + \alpha_m D_n k^2)^2 + \alpha_m^2\beta}{\omega_0^2\beta}\right] \tag{29}$$

m=1,2; n=3-m and ξ, τ represent the space lag and time lag respectively

Here obviously

$$\omega_0^2 - \frac{\beta^2}{4} > 0 \;\Rightarrow\; \varepsilon^2\gamma^4 + 1 < \varepsilon\gamma^2(\varepsilon\gamma+1) \tag{30}$$

The auto correlation functions (ACF) can be calculated numerically so that the inequality (30) is satisfied. We calculate the ACF by taking $\varepsilon = 10$, $\gamma = 0.5$, $D_1 = D_2 = 1$, the graphs of which are given in figures. From Fig. I, we see that both the populations exhibit non-cyclic fluctuations with respect to space.Fig. II exhibit interesting behavior with regard to fluctuation. Inspite of different growth rate equations, both the populations exhibit similar patterns of fluctuations. The ups down of fluctuations persists up to the time lag $\tau = 4.0$; after this value the fluctuation convariance of both the populations exhibit uniform pattern, being independent of the time lag.

ACKNOWLEDGEMENT

The author wishes to thank Council of scitific and Industrial Research, India for his Senior Research Fellowship. The author also wishes to thank the learned referee for his valuable comment, which has helped in the better exposition of the paper.

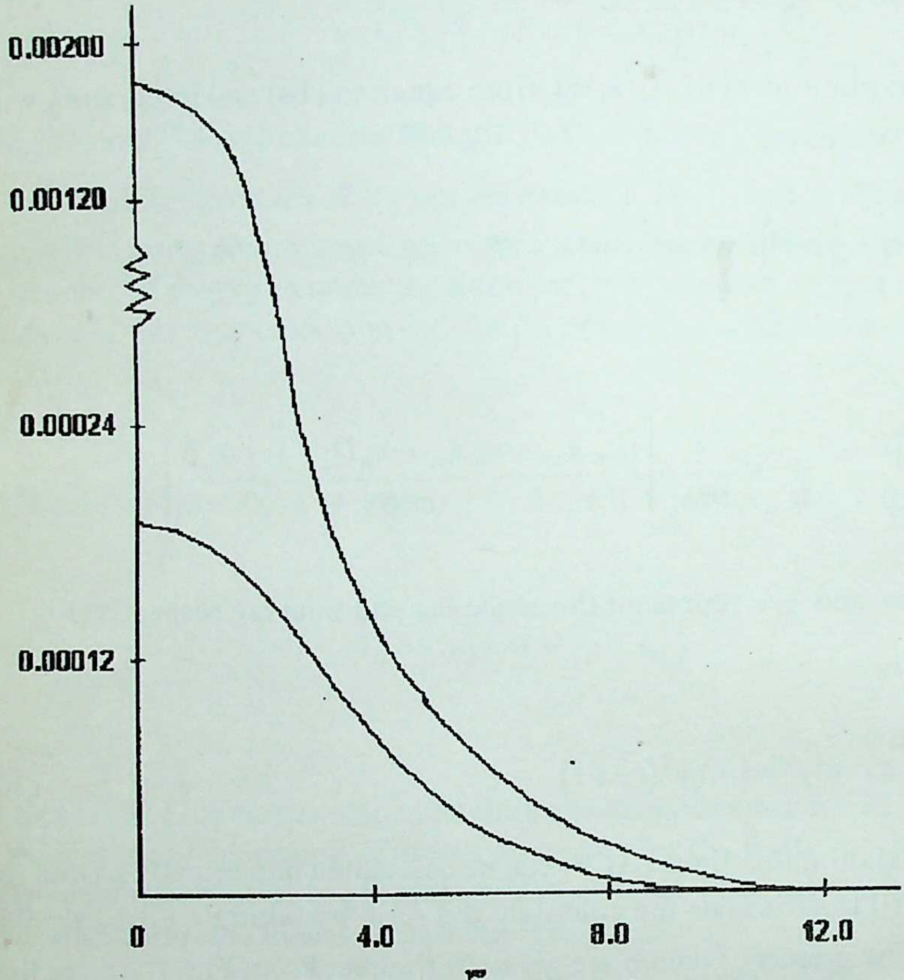

Fig. I

Fig. I exhibits non-cyclic fluctuations with respect to space of both the populations respectively.

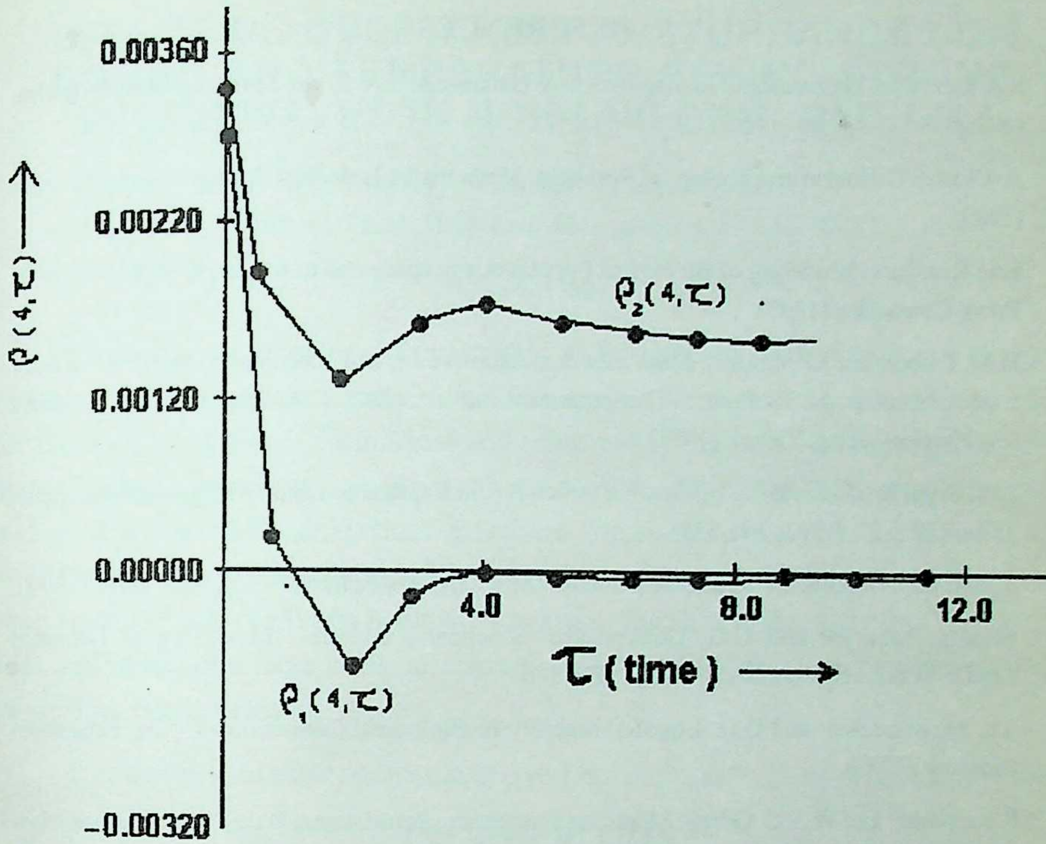

Fig. II exhibits non-cyclic fluctuations with respect to time for both the populations respectively.

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Journal of Natural & Physical Sciences, Vol. 14 (1-2) (2000) 49-58

EFFECT OF OVERTAKING DISTURBANCES ON SOUND AND TEMPERATURE BEHIND STRONG SHOCK WAVES IN NON-UNIFORM MEDIUM-I

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(Received 23.11.1996 and after revision 27.06.2000)

ABSTRACT

Considering the effect of overtaking disturbances, the propagation of sound and temperature behind strong shock in a non-uniform medium has been studied for three cases of plane, cylindrical and spherical symmetries. Assuming the initial density distribution law $\rho_0 = \rho' r^w$, where ρ' is the density at the axis of symmetry and w is a constant, analytical relations for sound velocity and temperature modified by overtaking disturbances have been obtained. An approximate method proposed by Yadav[13] has been used to study the shock propagation. The results accomplished here have been compared with those for freely propagating shock as well as earlier results.

It is found that the conclusion arrived here agrees with experimental results.

INTRODUCTION

Study of propagation of shock waves through different medium is of immense importance for the production of very high temperature and high pressure. The problem of strong shock waves has been studied by many authors[1-7], and many others. Among them, Yadav and Rana[8] have used CCW[9-11] method, to study the propagation of sound and temperature behind the strong shock in a uniform medium for three cases of plane, cylindrical and spherical symmetries. In the CCW approximation shock is not affected by the disturbances in the flow behind the shock. The effect of overtaking disturbances on the propagation of shock has been studied by Yousaf[12] and Yadav[13].

In the very recent communication, Yadav and Rana [14] studied the effect of overtaking disturbances behind strong shock in a uniform medium. In the present

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paper, the effect of overtaking disturbances on the propagation of shock and consequently the variation of sound and temperature behind strong shock has been studied in the non-uniform medium. Yadav[13] technique has been used to study the propagation of shock wave in a non-uniform medium having denstity distribution law as $\rho_0 = \rho' r^w$, where ρ' is the density at the centre/axis of symmetry. Analytical relations for non-dimensional sound velocity and temperature behind the shock have been obtained for plane, cylindrical and spherical symmetries.

Finally, the results obtained here are compared with those for freely propagating shock and well earlier results. It is observed that the conclusions arrived here are in agreement with experimental results (Terao and Wagner)[15].

THEORY

When a shock produced by an intense explosion moves in a medium, the disturbances are generated in the form of shock and travel along C_- Characteristics. These C-disturbances are reflected from the sonic line which exists behind the shock front and propagates along C_+ characteristics and finaly overtake the shock. Thus affects the propagation of shocks .

Under the assumption that the gas is inviscid and non-conducting of heat, the basic equations for one-dimensional adiabatic flow of a perfect gas enclosed by shock front are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

$$\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r}\right) + \rho a^2\left(\frac{\partial u}{\partial r} + \frac{\alpha u}{r}\right) = 0 \qquad (1)$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)(P\rho^{-\gamma}) = 0$$

Where, u, p, ρ denote respectively the particle velocity, the pressure and the density at a distance r from the origin at time t. The coefficient α takes the values 0,1, 2 respectively for plane, cylindrical and spherical symmetries of the

problem and γ is the adiabatic index of the gas.

Let P_0 and ρ_0 denote the undisturebed values of pressure and density in front of the shock and u_1, p_1 and ρ_1 be the values of the respective quantities at any point immediately after passage of the shock, then the well known Rankine-Hugoniot relations, permit us to express u_1, p_1 and ρ_1 in-terms of the undisturbed values of these quantities by mean of following equations:

$$P_1 = \rho_0 a_0^2 \left[\frac{2M^2}{\gamma+1} - \frac{(\gamma-1)}{\gamma(\gamma+1)} \right]$$

$$\rho_1 = \frac{\rho_0(\gamma+1)M^2}{2+(\gamma-1)M^2} \tag{2}$$

$$u_1 = \frac{2a_0(M^2-1)}{M(\gamma+1)}$$

$$U = a_0 M$$

Where, U is the shock velocity, a_0 is sound velocity undisturbed medium and M is Mach number. For strong shock these relation reduced to:

$$P = \frac{2\rho_0 U^2}{(\gamma+1)}, \quad \rho = \rho_0 \frac{\gamma+1}{\gamma-1}, \quad u = \frac{2U}{(\gamma+1)} \tag{3}$$

$$a = \frac{sU(\gamma-1)}{(\gamma+1)}, \quad T = T_0 \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} \left[\frac{U}{a_0} \right]^2 \tag{4}$$

Where, a is sound velocity, T is temperature behind the shock and T_0 is the temperature in undisturbed medium and

$$s = \left[\frac{2\gamma}{\gamma-1} \right]^{\frac{1}{2}}$$

THE FREELY PROPAGATION OF SHOCKS

Ignoring the effect of overtaking disturbances the characteristics from of system of equations (1) is taken as

$$dp + \rho\, adu + \frac{\alpha \rho a^2 u}{u+a} \cdot \frac{dr}{r} = 0 \tag{5}$$

a differential relation valid for C_- disturbances. Now using (3), (4) and (5) we have:

$$\frac{dU}{U} + \frac{w}{2+s}\frac{dr}{r} + \frac{\alpha s^2(\gamma-1)}{(2+s)\{2+s(\gamma-1)\}} \cdot \frac{dr}{r} = 0 \tag{6}$$

Here, it is assumed that $\rho_0 = \rho' r^w$, where ρ' is the density at the centre/ axis of symmetry and w is a constant. On integration (6), we have

$$U = K\, r^{-\left(\alpha A + \frac{w}{2+s}\right)} \tag{7}$$

$$\frac{U}{a_0} = K'\gamma^{-1/2} r^{-(\alpha A + \frac{w}{2+s})} \tag{8}$$

Where $K' = K(\rho_0 / P_0)^{1/2}$ and $a_0 = (\gamma P_0 / \rho_0)^{1/2}$ and $A = s^2(\gamma-1)/(2+s)\{2+s(\gamma+1)\}$

It is important to note further that a_0 remains constant even after passes of the shock. Now relations for non-dimensional sound velocity and temperature behind shock can be written as :

$$\frac{a}{a_0} = \frac{s(\gamma-1)}{(\gamma+1)} K'\gamma^{-1/2} r^{-[\alpha A + w/(2+s)]} \tag{9}$$

and

$$\frac{T}{T_0} = \frac{2(\gamma-1)K'^2}{(\gamma+1)^2} r^{-2[\alpha A + w/(2+s)]} \tag{10}$$

THE OVERTAKING DISTURBANCES

In this section we consider the effect of overtaking disturbances on the motion of shock waves. It is assumed that the shock propagates along C_- characteristics and produces the pressure and fluid velocity in increment, dp_- and du_- respectively, while overtaking disturbances, propagate along C_+ characteristics creat dp_+ and du_+. Here it is assumed increments in pressure and fluid velocity are supported both by C_- and C_+ disturbances.

To estimate the strength of overtaking disturbances an independent C_+ characteristics is considered.The differential relation valid across C_+ characteristics is taken as:

$$dp - \rho a du + \frac{\alpha \rho a^2 U}{(u-a)} \cdot \frac{dr}{r} = 0 \tag{11}$$

The next step is to substitute the shock conditions (3), (4) into this relation we get:

$$\frac{dU}{U} + \left[\frac{w}{2-s} + \frac{\alpha s^2(\gamma-1)}{(2-s)\{2-s(\gamma-1)\}}\right] \frac{dr}{r} = 0 \tag{12}$$

Using,(3), (4) and (12) we have:

$$du_+ = -\frac{2U}{(\gamma+1)(2-s)}\left[w + \frac{\alpha s^2(\gamma-1)}{(2-s)\{2-s(\gamma-1)\}}\right] \frac{dr}{r} = 0 \tag{13}$$

which determine the strength of the overtaking disturbances. For C_- disturbances, the fluid velocity increment, by use of (4) and (6) may be expressed as:

$$du_- = -\frac{2U}{(\gamma+1)(2-s)}\left[w + \frac{\alpha s^2(\gamma-1)}{[2+s(\gamma-1)]}\right] \frac{dr}{r} = 0 \tag{14}$$

The expression (7) for freely propagating shock is obtained by requiring that the pressure

and fluid velocity increments behind the shock by supported solely by C_- disturbances. In consideration of overtaking disturbances i.e. in presence of C_- and C_+ disturbances, the fluid velecity increment will be related as:

$$du_- + du_+ = \frac{2}{(\gamma+1)} dU \tag{15}$$

substituting value from (13) and (14) into this relation, we get after simplification:

$$U^* = Kr^{-\left[\alpha B + 4w/\left(4-s^2\right)\right]}$$

$$\frac{U^*}{a_0} = K'\gamma^{-\frac{1}{2}} r^{-\left[\alpha B + 4w/\left(4-s^2\right)\right]} \tag{16}$$

This expression represent the shock velecity modified by overtaking distrubances. Modified expression for sound velecity and temperature can be written as :

$$\frac{a^*}{a_0} = \frac{s(\gamma-1)}{(\gamma+1)} K'\gamma^{-1/2} r^{-[\alpha B + 4w/(4-s^2)]} \tag{17}$$

and

$$\frac{T^*}{T_0} = \frac{2(\gamma-1)}{(\gamma+1)^2} K'^2 r^{-[\alpha B + 4w/(4-s^2)]} \tag{18}$$

where

$B = 4\gamma(2+\gamma)/(4-s^2)[2-\gamma(\gamma-1)]$

DISCUSSIONS

Equation (8), (9) (10) and (16),(17), (18) respectively, represent non- dimensional shock velecity , sound velecity, temperature for freely propagation shock and under the influence of overtaking disturbances. For initial density distribution $\rho_0 \propto r^{0.1}$ and $U/a_0 = 10$ at $r = 20$ for $\gamma = 1.2$ the variation of shock velocity, sound velocity and temperature with propagation distance r for spheri-

cal, cylindrical and plane symmetries have been given in the TableI(a), I(b) & I (c). It is found that shock velocity, sound velocity and temperature decay asymptotically in case of freely propagating shock, while in the presence of overtaking disturbances, these parameters increase asymptotically with propagating distance for spherical and cylindrical cases.For plane shock these parameters increase in freely propagating shock and decrease with propagating distance r in case of overtaking disturbances.

It is important to mention that the temperature variation is very faster in presence of overtaking disturbances for spherical and cylindrical shock (Table I (a) I (b)), This agrees with experimental result of Terao and Wanger[15].

The variation of sound velecity and temperature with w is shown in the Table II(a), II(b), II(c) respectively for spherical cylindrical and plane symmetries. It is observed that sound velocity and temperature decrease with the increase of w is freely propagating shock. In case of overtaking disturbances sound velocity and temperature increase with w for all the symmetries. Dependence of shock velocity, sound velocity and temperature on w is usefull to prepare the density profile of the medium.

The variation of non-dimensional sound velocity and temperature with γ has been shown in Table III(a), III(b), III(c). An increase in sound velocity and temperature with γ behind strong shock for both the cases (i.e. freely propagation and with overtaking disturbances) is thermodynamically as well as astrophysically important).

ACKNOWLEDGEMENT

The first author (RPY) thanks the University Grants Commission of India for financial support and third author (A. Singh) thanks to Dr. R. P. Singh, Principal , Bareilly College, Bareilly for providing research facilities.

Comparison of non-dimensional shock velocity (u/a_0) sound veloaty (a/a_0), (T/T_0) in case of freely propagating shock and (u^*/a_0) and (T^*/T_0) with overtaking disturbances. The variation of these values with propagation distance are also given below

Table I (a) For strong spherical shock at $\gamma = 1.2$ $w = -0.1$

r	U/a_0	U^*/a_0	a/a_0	a^*/a_0	T/T_0	T^*/T_0
10	12.38	3407.12	3.90	1072.96	15.20	1151251.4
11	12.02	4174.73	3.79	1314.70	14.33	1728438.9
12	11.70	5025.57	3.69	1582.65	13.58	2504766.9
13	11.42	5960.62	3.60	1877.11	12.93	3523538.6
14	11.16	6980.76	3.51	2198.37	12.35	4832831.5
15	10.93	8086.83	3.44	2546.69	11.84	6458640.2
16	10.71	9279.62	3.37	2922.32	11.38	8539965.6
17	10.51	10559.86	3.31	3225.49	10.96	11058906.3
18	10.33	11928.26	3.25	3756.43	10.58	14110747.4
19	10.16	13385.48	3.19	4215.33	10.24	17769042.5
20	10.00	14932.17	3.15	4702.41	9.91	22112694.7

(b) For strong spherical shock at $\gamma = 1.2$ $w = -0.1$

r	U/a_0	U^*/a_0	a/a_0	a^*/a_0	T/T_0	T^*/T_0
10	18.07	276.37	5.68	87.03	32.08	7574.78
11	17.77	305.19	5.60	96.11	31.33	9237.22
12	17.55	334.12	5.53	105.22	30.55	11071.58
13	17.35	363.15	5.46	114.36	29.85	13079.09
14	17.16	392.27	5.40	123.53	29.22	15260.88
15	16.99	421.48	5.35	132.73	28.64	17618.01
16	16.84	450.77	5.30	191.91	28.10	20151.49
17	16.69	480.13	5.25	151.20	27.62	22862.24
18	16.55	509.56	5.21	160.47	27.16	25751.14
19	16.42	539.06	5.17	169.76	26.74	28819.03
20	16.30	567.82	5.13	179.07	26.35	32066.71

(c) For strong spherical shock at $\gamma = 1.2$ $w = -0.1$

r	U/a_0	U^*/a_0	a/a_0	a^*/a_0	T/T_0	T^*/T_0
10	26.24	22.42	8.26	7.06	68.26	49.84
11	26.28	22.31	8.28	7.03	68.50	49.37
12	26.32	22.21	8.29	6.99	68.72	48.94
13	26.36	22.12	8.30	6.97	68.92	48.55
14	26.40	22.04	8.31	6.94	69.10	48.19
15	26.43	21.97	8.32	6.92	69.28	47.86
16	26.46	21.90	8.33	6.90	69.44	47.26
17	26.49	21.83	8.34	6.88	69.60	47.26
18	26.52	21.77	8.35	8.86	69.75	46.99
19	26.55	21.71	8.36	6.84	69.88	46.74
20	26.57	21.65	8.37	6.82	70.01	46.50

The variation of non-dimensional shock velocity (U/a_0) sound velocity (a/a_0) temperature T/T_0 in freely propagation and (U^*/a_0) (T^*/T_0) with w are shown below

Table II (a) For strong spherical shock at $r = 10$, $\gamma = 1.2$

w	U/a_0	U^*/a_0	a/a_0	a^*/a_0	T/T_0	T^*/T_0
+0.1	11.38	4289.30	3.58	1350.78	12.84	1824610.3
0	11.87	3822.84	3.74	1203.88	13.97	1449339.5
-0.1	12.38	3407.12	3.90	1072.96	15.20	1151251.4

(b) For strong spherical shock at $r = 10$, $\gamma = 1.2$

w	U/a_0	U^*/a_0	a/a_0	a^*/a_0	T/T_0	T^*/T_0
+0.1	16.56	347.93	5.21	109.57	27.21	12005.22
0	17.28	310.09	5.44	97.65	29.61	9536.08
-0.1	18.02	276.37	5.68	87.03	32.08	7574.75

(c) For strong spherical shock at $r = 10$, $\gamma = 1.2$

w	U/a_0	U^*/a_0	a/a_0	a^*/a_0	T/T_0	T^*/T_0
+0.1	24.12	28.22	7.59	8.88	57.67	78.99
0	25.15	25.15	7.92	7.92	62.92	62.92
-0.1	26.24	22.42	8.26	7.06	68.26	49.84

The variation of non-dimensional shock velocity (U/a_0), sound velocity (a/a_0) temperature (T/T_0) in-fereely propagation and (U^*/a_0), (a^*/a_0), (T^*/T_0) in case of ovetaking disturbances with adiabatic gas constant γ given below.

Table III (a) For strong spherical shock at $r = 10$, $w = -0.1$

γ	U/a_0	U^*/a_0	a/a_0	a^*/a_0	T/T_0	T^*/T_0
6/5	12.38	2407.12	3.90	1072.96	15.10	115125.40
11/9	12.01	8780.16	3.97	2960.05	15.87	8480042.10
5/4	11.59	34262.79	4.07	12038.72	16.58	144930740.0

(b) For strong spherical shock at $r = 10$, $w = -0.1$

γ	U/a_0	U^*/a_0	a/a_0	a^*/a_0	T/T_0	T^*/T_0
6/5	18.02	276.37	5.68	87.03	32.08	7574.78
11/9	17.68	437.93	5.86	147.67	34.40	21095.90
5/4	17.28	850.99	6.07	299.01	36.88	89404.73

(c) For strong spherical shock at $r = 10$, $w = -0.1$

γ	U/a_0	U^*/a_0	a/a_0	a^*/a_0	T/T_0	T^*/T_0
6/5	26.24	22.42	8.26	7.06	68.26	49.84
11/9	26.03	21.85	8.63	7.25	74.51	52.52
5/4	25.77	21.13	9.05	7.42	81.98	55.16

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Journal of Natural & Physical Science, Vol. 14 (1-2) (2000) 59-70

DARBOUX PROBLEM FOR HYPERBOLIC PARTIAL INTEGRODIFFERENTIAL EQUATIONS

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(Received 17.07.2000)

ABSTRACT

In this paper we obtain sufficient conditions for the existence, uniqueness and continous dependence on parameter of solutions of Darboux problem for hyperbolic partial integrodifferential equations in two independent variables. The well known Banach fixed point theorem coupled with the Bilecki type norm is used to establish the results. Applications to Darboux problem for hyperbolic partial integrodifferential equations involving more than two independent variables are also indicated.

INTRODUCTION

Consider the Darboux problem for the hyperbolic partial integrodifferential equation of the form

$$u_{xy} = f\left(x, y, u, u_x, u_y, \int_0^x\int_0^y g(x, y, s, t, u, u_s, u_t)dsdt, \lambda\right) \tag{1}$$

with the given conditions

$$u(x, o) = \sigma(x) + \tau(o),\ u(o, y) = \tau(y) + \sigma(o), \tag{2}$$

where u is an unknown function of two variables and λ is a real parameter, $g \in C[\Delta^2 \mathrm{x} E^3, E], f \in C[\Delta \mathrm{x} E^4 \mathrm{x} R, E]$, $\sigma \in C^1[(o,\infty), E]$, $\tau \in C^1[(o,\infty), E]$, in which E denotes the n-dimensional Euclidan space with convenient norm $\|\cdot\|$. $\Delta = [0,\infty)\mathrm{x}[0,\infty)$, $\Delta^2 = \{(x, y, s.t): 0 \le s \le x < \infty,\ 0 \le t \le y < \infty\}$, $R = (-\infty, +\infty)$.

In [11] B. Palczewski has studied the uniqueness and the convergence of successive approximations of the problem (1)-(2) when $\lambda = 0$. The results established in [11] impose certain conditions which are further generalizations of the conditions used by F. Brauer in [3]. The questions of existence and uniqueness of solutions of special forms of problem (1)-(2) when the integral

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term in (1) is absent and $\lambda = 0$ have been studied by many authors under a variety of hypotheses by using different techniques, see [1, 7, 13-15].

The purpose of this paper is to present sufficient conditions which guarantee the existence and uniqueness of the solutions of problem (1)-(2). We also determine conditions for continuous dependence of solutions of problem (1)-(2) on a parameter λ. Our results are obtained by using the well known Banach fixed point theorem. The method employed in our analysis can be used to study the Darboux problem for partial hyperbolic integrodifferential equations involving more than two independent variables. The sufficient conditions obtained here are different from those obtained by B. Palczewski in [11] and the results presented in this paper are derived in a rather simple and unified way, which makes further applications to more general Darboux problems easier.

MAIN RESULTS

In this section we present our results on the existence, uniqueness and continuous dependence on parameter of solutions of problem (1)-(2). As in [11] we say that the function $u(x,y)$ with values in E belongs to the class $C^*(\Delta)$, if it is continuous on Δ together with its partial derivatives u_x, u_y and u_{xy} with values in E. By a solution of problem (1)-(2) we mean a function $u(x,y) \in C^*(\Delta)$ fulfilling equation (1) and conditions in (2).

For convenience we list the following hypotheses used in our subsequent discussion.

(H_1) For $\lambda \in R$, (x,y,s,t,u_1,u_2,u_3), $(x,y,s,t,\bar{u}_1,\bar{u}_2,\bar{u}_3) \in \Delta^2 \times E^3$,

$(x,y,u_1,u_2,u_3,v,\lambda)$, $(x,y,\bar{u}_1,\bar{u}_2,\bar{u}_3,v,\lambda) \in \Delta \times E^4 \times R$,

$$\|g(x,y,s,t,u_1,u_2,u_3) - g(x,y,s,t,\bar{u}_1,\bar{u}_2,\bar{u}_3)\|$$

$$\le M[\|u_1-\bar{u}_1\| + \|u_2-\bar{u}_2\| + \|u_3-\bar{u}_3\|]$$

$$\|f(x,y,u_1,u_2,u_3,v,\lambda) - f(x,y,u_1,u_2,u_3,\bar{v},\lambda)\|$$

$$\le L[\|u_1-\bar{u}_1\| + \|u_2-\bar{u}_2\| + \|u_3-\bar{u}_3\| + \|v-\bar{v}\|],$$

where L and M are nonnegative constants.

(H_2) Let μ be a positive constant. For every fixed $\lambda \in R$ there exist nonnegative constants P_i (i=1,2,3) such that

$$\|\sigma(x)\| + \|\tau(y)\| + \int_0^x \int_0^y \left\| f\left(s,t,0,0,0,\int_0^s \int_0^t g(s,t,\xi,\eta,0,0,0)d\xi d\eta,\lambda\right)\right\| dsdt$$

$$\leq P_1 \exp(\mu(x+y)),$$

$$\|\sigma_x(x)\| + \int_0^y \left\| f\left(x,t,0,0,0,\int_0^x \int_0^t g(x,t,\xi,\eta,0,0,0)d\xi d\eta,\lambda\right)\right\| dt$$

$$\leq P_2 \exp(\mu(x+y)),$$

$$\|\tau_y(y)\| + \int_0^x \left\| f\left(s,y,0,0,0,\int_0^s \int_0^y g(s,y,\xi,\eta,0,0,0)d\xi d\eta,\lambda\right)\right\| ds$$

$$\leq P_3 \exp(\mu(x+y)),$$

where 0 is the zero element in E.

(H_3) There exist nonnegative constants N_i (i=1,2,3) and a function

$A(x,y) : \Delta \to R^+ = [0,\infty)$ such that

$$\|f(x,y,u_1,u_2,u_3,v,\lambda_1) - f(x,y,u_1,u_2,u_3,v,\lambda_2)\|$$

$$\leq A(x,y)|\lambda_1 - \lambda_2|,$$

for $(x,y,u_1,u_2,u_3,v,\lambda_i) \in \Delta \times E^4 \times R$ $(i = 1,2)$ and

$$\int_0^x \int_0^y A(s,t)dsdt \leq N_1 \exp(\mu(x+y)),$$

$$\int_0^y A(x,t)dt \leq N_2 \exp(\mu(x+y)),$$

$$\int_0^x A(s,y)ds \le N_3 \exp(\mu(x+y)),$$

where $|r|$ denotes the absolute value of $r \in R$.

For every function $z \in C^*(\Delta)$, we denote

$$\|z(x,y)\|_1 = \|z(x,y)\| + \|z_x(x,y)\| + \|z_y(x,y)\|,\ (x,y) \in \Delta \tag{3}$$

Let B be a space of those functions $z \in C^*(\Delta)$, which fulfil the condition

$$\|z(x,y)\|_1 = 0(\exp(\mu(x+y))),\ (x,y) \in \Delta,$$

where $\mu > 0$ is a constant. In the space B we define the norm (see [2 and also 5,9,10]).

$$\|z\|_B = \sup_{(x,y)\in\Delta}\left[\|z(x,y)\|_1 \exp(-\mu(x+y))\right]. \tag{4}$$

It is easy to see that B with norm defined in (4) is a Banach space (see [5]. We note that the condition (3) implies that there exists a constant $M_0 \ge 0$ such that $\|z(x,y)\|_1 \le M_0 \exp(\mu(x+y)),\ (x,y) \in \Delta$. Using this fact in (4) we observe that

$$\|z\|_B \le M_0. \tag{5}$$

We now establish our main result on the existence and uniqueness of solutions of problem (1)-(2).

Theorem 1. *Suppose that the hypotheses* (H_1), (H_2) *hold. If* $\alpha = L\left(1+\frac{M}{\mu^2}\right)\left(\frac{1}{\mu^2}+\frac{2}{\mu}\right) < 1$ *then there exists a unique solution* $u \in B$ *of the problem* (1)-(2).

Proof. Let $\lambda \in R$ be fixed. For $u \in B$, we define the operator $(T_\lambda u)$ by

$$(T_\lambda u)(x,y) = \sigma(x) + \tau(y) + \int_0^x\int_0^y f(s,t,u(s,t),u_s(s,t),u_t(s,t),$$

$$\int_0^s\int_0^t g\left(s,t,\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta),u_\eta(\xi,\eta)d\xi d\eta,\lambda\right)dsdt. \tag{6}$$

Clearly the solution of problem (1)-(2) is a fixed point of the operator equation

$$u=(T_\lambda u). \tag{7}$$

Now we shall prove that T_λ maps B into itself. From (6) and hypotheses (H_1), (H_2) we obtain

$$\|(T_\lambda u)(x,y)\| \le \|\sigma(x)\|+\|\tau(y)\|$$

$$+\int_0^x\int_0^y \Big\| f\big(s,t,u(s,t),u_s(s,t),u_t(s,t),$$

$$\int_0^s\int_0^t g\left(s,t,\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta),u_\eta(\xi,\eta)\right)d\xi d\eta,\lambda\big)$$

$$-f\left(s,t,0,0,0,\int_0^s\int_0^t g(s,t,\xi,\eta,0,0,0)d\xi d\eta,\ \lambda\right)\Big\|dsdt$$

$$+f\int_0^x\int_0^y\left\|f\left(s,t,0,0,0,\int_0^s\int_0^t g(s,t,\xi,\eta,0,0,0)d\xi d\eta,\lambda\right)\right\|dsdt$$

$$\le \|u\|_B\int_0^x\int_0^y L\left[\exp(\mu(s+t))+\int_0^s\int_0^t M\exp(\mu(\xi+\eta)d\xi d\eta)\right]dsdt+P_1\exp(\mu(x+y))$$

$$\le L\left(1+\frac{M}{\mu^2}\right)\frac{1}{\mu^2}\|u\|_B\exp(\mu(x+y))+P_1\exp(\mu(x+y)). \tag{8}$$

Differentiating (6) with respect to x and using hypotheses (H_1), (H_2) we obtain

$$\|(T_\lambda u)_x(x,y)\| \le \|\sigma_x(x)\|+\int_0^y\Big\|f\big(x,t,u(x,t),u_x(x,t),u_t(x,t),$$

$$\int_0^x\int_0^t g\left(x,t,\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta),u_\eta(\xi,\eta)\right)d\xi d\eta,\lambda\big)\Big\|$$

$$-f\left(x,t,0,0,0,\int_0^x\int_0^t g(x,t,\xi,\eta,0,0,0)d\xi d\eta,\lambda\right)\Bigg\|dt$$

$$+\int_0^y\left\|f\left(x,t,\xi,\eta,0,0,0,\int_0^x\int_0^t g(x,t,\xi,\eta,0,0,0)d\xi d\eta,\lambda\right)\right\|dt$$

$$\leq\|u\|_B\int_0^y L\left[\exp(\mu(x+t))+\int_0^x\int_0^t M\exp(\mu(\xi+\eta))d\xi d\eta\right]dt+P_2\exp(\mu(x+y))$$

$$\leq L\left(1+\frac{M}{\mu^2}\right)\frac{1}{\mu}\|u\|_B\exp(\mu(x+y))+P_2\exp(\mu(x+y)). \tag{9}$$

Similarly, differentiating (6) with respect to y and using hypotheses (H_1), (H_2) we obtain

$$\|(T_\lambda u)_y(x,y)\|\leq L\left(1+\frac{M}{\mu^2}\right)\frac{1}{\mu}\|u\|_B\exp(\mu(x+y))+P_3\exp(\mu(x+y)). \tag{10}$$

From (8) -(10) and (5) we observe that

$$\|(T_\lambda u)(x,y)\|_1\leq\left[L\left(1+\frac{M}{\mu^2}\right)\left(\frac{1}{\mu^2}+\frac{2}{\mu}\right)M_0+P_1+P_2+P_3\right]\exp(\mu(x+y)).$$

This shows that T_λ maps B into itself. Now we verify that the operator T_λ defined in (6) is a contraction map. Let $u,\ \bar{u}\in B$. From (6) and hypothesis (H_1) we obtain

$$\|(T_\lambda u)(x,y)-(T_\lambda\bar{u})(x,y)\|$$

$$\leq\int_0^x\int_0^y\|f(s,t,u(s,t),u_s(s,t),u_t(s,t),$$

$$\int_0^s\int_0^t g(s,t,\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta),u_\eta(\xi,\eta)d\xi d\eta,\lambda)$$

$$-f(s,t,\bar{u}(s,t),\bar{u}_s(s,t),\bar{u}_t(s,t),$$

$$\int_0^s\int_0^t g\left(s,t,\xi,\eta,\bar{u}(\xi,\eta),\bar{u}_\xi(\xi,\eta),\bar{u}_\eta(\xi,\eta)\right)d\xi d\eta,\lambda)\|dsdt$$

$$\leq \|u-\bar{u}\|_B \int_0^x\int_0^y L\left[\exp(\mu(s+t)) + \int_0^s\int_0^t M\exp(\mu(\xi+\eta))d\xi d\eta\right]dsdt$$

$$\leq L\left(1+\frac{M}{\mu^2}\right)\frac{1}{\mu^2}\|u-\bar{u}\|_B \exp(\mu(x+y\}). \tag{11}$$

Differentiating (6) with respect to x and using hypothesis (H_1) we obtain

$$\left\|(T_\lambda u)_x(x,y)-(T_\lambda \bar{u})_x(x,y)\right\|$$

$$\leq \int_0^y \left\|f\left(x,t,u(x,t),u_x(x,t),u_t(x,t),\right.\right.$$

$$\int_0^x\int_0^t g\left(x,t,\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta),u_\eta(\xi,\eta)\right)d\xi d\eta,\lambda)$$

$$-f\left(x,t,\bar{u}(x,t),\bar{u}_x(x,t),\bar{u}_t(x,t)\right.,$$

$$\int_0^x\int_0^t g\left(x,t,\xi,\eta,\bar{u}(\xi,\eta),\bar{u}_\xi(\xi,\eta),u_\eta(\xi,\eta)\right)d_\xi d\eta,\lambda)dt$$

$$\leq \|u-\bar{u}\|_B \int_0^y L\left[\exp(\mu(x+t)) + \int_0^x\int_0^t M\exp(\mu(\xi+\eta))d_\xi d\eta\right]dt$$

$$\leq L\left(1+\frac{M}{\mu^2}\right)\frac{1}{\mu}\|u-\bar{u}\|_B \exp(\mu(x+y)). \tag{12}$$

Similarly, differentiating (6) with respect to y and using hypothesis (H_1) we obtain

$$\left\|(T_\lambda u)_y(x,y)-(T_\lambda \bar{u})_y(x,y)\right\|$$

$$\leq L\left(1+\frac{M}{\mu^2}\right)\frac{1}{\mu}\|u-\bar{u}\|_B \exp(\mu(x+y)). \tag{13}$$

From (11)-(13) we obtain

$$\|(T_\lambda u)-(T_\lambda \bar{u})\|_B \leq \alpha\|u-\bar{u}\|_B . \tag{14}$$

Since $\alpha<1$, it follows from Banach fixed point theorem [8] that T_λ has a unique fixed point in B. The fixed point of T_λ is however a solution of problem (1)-(2) and the proof is complete.

As an immediate consequence of Theorem 1 we obtain the following.

Theorem 2. Assume that the conditions of Theorem 1 hold. Then for any $u^0 \in B$ the sequence $\{u^{(k)}\}$ given successively by

$$u^{(k)}(x,y)=\sigma(x)+\tau(y)+\int_0^x\int_0^y f\left(s,t,u^{(k-1)}(s,t),u_s^{(k-1)}(s,t),u_t^{(k-1)}(s,t),\right.$$

$$\left.\int_0^s\int_0^t g\left(s,t,\xi,\eta,u^{(k-1)}(\xi,n),u_\xi^{(k-1)}(\xi,\eta),u_\eta^{((k-1))}(\xi,\eta)\right)d\xi d\eta,\lambda\right)dsdt \tag{15}$$

for k=1,2 ..., converges in B to a unique solution $u(x,y)$ of problem (1)-(2). Moreover

$$\left\|u^{(k)}-u^{(0)}\right\|_B \leq \frac{\alpha^n}{1-\alpha}\left\|u^{(1)}-u^{(0)}\right\|_B, \tag{16}$$

for k= 1,2,..., where α is as defined in Theorem 1.

Proof. Let $u^{(0)}(x,y) \in B$ be given. Then we can determine a sequence $\{u^{(k)}\}$ successively from

$$u^{(k)}=\left(T_\lambda u^{(k-1)}\right),\ k=1,2,.... \tag{17}$$

It is easy to observe form Theorem 1 that the sequene $\{u^{(k)}\}$ determined from (17)

converges to a unique solution $u \in B$ of (7). Since (7) and (17) are the operator equations of problem (1)-(2) and (15) respectively, we conclude that the sequence $\{u^{(k)}\}$ given by (15) converges in B to the unique solution u of problem (1)-(2). Finally, the error estimate (16) follows from the contraction property of T_λ and the proof of the theorem is complete.

We next establish the following theorem which determine conditions for continuous dependence of solutions of problem (1)-(2) on a parameter λ. The idea of the proof is based on the result given by the present author in [9, Theorem 2] (see also [5,10]).

Theorem 3. Let the conditions in Theorem 1 holds. In addition, assume that the hypothesis (H_3) holds. Then the solution $u(x, y, \lambda)$ of problem (1)-(2) belonging to B is continous with respect to the variables (x, y, λ) in $\Delta \times R$.

Proof. For $u \in B$ and $\lambda \in R$, define the operator (T_λ u) by (6). From (6) and hypothesis (H_3) we have

$$\left\|(T_\lambda u)(x,y)-\left(T_{\lambda_0}u\right)(x,y)\right\| \le \int_0^x\int_0^y A(s,t)|\lambda-\lambda_0|dsdt$$

$$\le |\lambda-\lambda_0|N_1\exp(\mu(x+y)). \tag{18}$$

Differentiating (6) with respect to x and using hypothesis (H_3) we obtain

$$\left\|(T_\lambda u)_x(x,y)-\left(T_{\lambda_0}u\right)_x(x,y)\right\| \le \int_0^y A(x,t)|\lambda-\lambda_0|dt$$

$$\le |\lambda-\lambda_0|N_2\exp(\mu(x+y)). \tag{19}$$

Similarly, differentiating (6) with respect to y and using hypothesis (H_3) we have

$$\left\|(T_\lambda u)_y(x,y)-\left(T_{\lambda_0}u\right)_y(x,y)\right\| \le |\lambda-\lambda_0|N_3\exp(\mu(x+y)). \tag{20}$$

From (18)-(20) we obtain

$$\left\|(T_\lambda u)-\left(T_{\lambda_0}u\right)\right\|_B \le (N_1+N_2+N_3)|\lambda-\lambda_0|. \tag{21}$$

From Theorem 1 there exists a unique solution $u(x,y,\lambda)$ of problem (1)-(2) such that

$(T_\lambda u)(x,y) = u(x,y,\lambda)$ *and* $(T_{\lambda_0} u)(x,y,\lambda_0) = u(x,y,\lambda_0)$ for $(x,y) \in \Delta$. Therefore from (14) and (21) we have

$$\begin{aligned}\left\|u(x,y,\lambda)-u(x,y,\lambda_0)\right\|_B &\le \left\|(T_\lambda u)(x,y,\lambda)-(T_\lambda u)(x,y,\lambda_0)\right\|_B \\ &\quad +\left\|(T_\lambda u)(x,y,\lambda_0)-(T_{\lambda_0} u)(x,y,\lambda_0)\right\|_B \\ &\le \alpha\left\|u(x,y,\lambda)-u(x,y,\lambda_0)\right\|_B \\ &\quad +(N_1+N_2+N_3)|\lambda-\lambda_0|.\end{aligned}$$

Hence

$$\left\|u(x,y,\lambda)-u(x,y,\lambda_0)\right\|_B \le (1-\alpha)^{-1}(N_1+N_2+N_3)|\lambda-\lambda_0|$$

This shows that the function $u(x,y,\lambda)$ is continous with respect to the variable λ in R uniformly with respect to the variables (x,y) in Δ and consequently $u(x,y,\lambda)$ is also continuous with respect to the variables (x,y,λ) in $\Delta \times R$, and the proof of the thorem is complete.

We note that the problem (1)-(2) is of more general type and contains as a special case (when $\lambda = 0$ and the integral term in (1) is absent) the Darboux problem studied by many authors in the literature (see, in particular [1,7,11,13-15] by using different techniques and assumptions. An important feature of our approach here is that it is elementary and provides a uniform treatment for such type of problems.

FURTHER APPLICATIONS

In this section, we indicate further applications of the method employed in Section 2 to study the existence, uniqueness and continuous dependence of solutions of Darboux problem for hyperbolic partial integrodifferentaial equations involving more than two indepenednt variables which in turn contains as special cases the Darboux problems studied by Palczewski [12], Conlan and Diaz [4], Glick [6] and several others.

In [12] B. palczewski has studied the existence and uniqueness of solutions of the Darboux problem for the equation

$$u_{x_1x_2x_3} = f\left(x_1,x_2,x_3,u,u_{x_1},u_{x_2},u_{x_3},u_{x_1x_2},u_{x_2x_3},u_{x_3x_1}\right) \tag{22}$$

under the conditions

$$u(0,x_2,x_3)=z_1(x_2,x_3),u(x_1,0,x_3)=z_2(x_1,x_3),u(x_1,x_2,0)=z_3(x_1,x_2), \tag{23}$$

by using the method of successive approximations and Schauder's fixed point theorem. We note that the results obtained in Section 2 can be very easily extended to the Darboux problem for partial hyperbolic integrodifferentaial equations of the form

$$u_{x_1x_2x_3}=f(x_1,x_2,x_3,u,u_{x_1},u_{x_2},u_{x_3},u_{x_1x_2},u_{x_2x_3},u_{x_3x_1},$$
$$\int_0^{x_1}\int_0^{x_2}\int_0^{x_3} g(x_1,x_2,x_3,s_1,s_2,s_3,u,u_{s_1},u_{s_2},u_{s_3},$$
$$u_{s_1s_2},u_{s_2s_3},u_{s_3s_1})ds_1ds_2ds_3,\lambda), \tag{24}$$

with the given conditions in (23). The problem (24)-(23) is a further generalization of the Darboux problem studied by palczewski in [12] by using different technique.

In [4,6] the authors have studied the existence and uniqueness of the solutions of hyperbolic partial differential equations of the form

$$u_{x_1\ldots x_n}=f(x_1,\ldots,x_n,u,u_{x_1},\ldots u_{x_n},u_{x_1x_2},\ldots,u_{x_nx_1},\ldots,$$
$$u_{x_1\ldots x_{n-1}},\ldots,u_{x_2\ldots x_n}) \tag{25}$$

where only the pure mixed derivatives of u appear, under the given boundary conditions. We note that one can very easily extend the results given in Section 2 to the Darboux problem for partial hyperbolic integrodifferential equations of the form

$$u_{x_1\ldots x_n}=f(x_1,\ldots,x_n,u,u_{x_1},\ldots u_{x_n},u_{x_1x_2},\ldots,u_{x_nx_1},\ldots,$$
$$u_{x_1\ldots x_{n-1}},\ldots,u_{x_2\ldots x_n},$$
$$\int_0^{x_1}\ldots\int_0^{x_n} g(x_1\ldots x_n,\ s_1,\ldots s_n,\ u,\ u_{s_1},\ldots u_{s_n},$$
$$u_{s_1s_2},\ldots u_{s_ns_1},\ldots,$$
$$u_{s_1\ldots s_{n-1}},\ldots,u_{s_2\ldots s_n})ds_1\ldots ds_n,\lambda), \tag{26}$$

under the given boundary conditions. The details of the formulation of such results are very close to those given in Section 2 under suitable modifications. We omit the details.

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Journal of Natural & Physical Science, Vol. 14 (1-2) (2000) 71-78

PERIODATE OXIDATION OF p-ETHYLANILINE IN ACETONE-WATER MEDIUM : A KINETIC AND MECHANISTIC STUDY

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(Received 20.04.2000 and after revision 24.07.2000)

ABSTRACT

The kinetics of oxidation of p-ethylaniline by periodate in acetone-water medium has been studied. The order with respect to both oxidant and substrate has been found to be one in each. The rate increases on increasing ionic strength while it decreases with a decrease in dielectric constant. The rate-pH profile has been given and discussed. There is no effect of free radical scavengers on rate of reaction. The thermodynamic parameters are also presented and discussed. The main product of oxidation characterised by UV-VIS, I.R. and N.M.R. spectrum was 4-ethy1-1, 2-benzoquinone. A suitable mechanism has been proposed and the rate law derived.

Keywords: Periodate oxidation, p-ethylaniline, kinetics and mechanism.

INTRODUCTION

In continuation of our studies on periodate oxidation[1,2], we are reporting in this paper the results of periodate oxidation of p-ethylaniline (PEA) in acetone- water medium.

EXPERIMENTAL

PEA and sodiummetperiodate of Merck A.R. grade were used after redistillation/ recrystallization respectively. All other chemicals used were of A.R. Grade. Doubly distilled water was used for the preparation of solutions and reaction mixtures. Thiel, Schultz and Coch buffer [4, 7] was used for maintaining the pH of solutions.

The reaction being quite fast at ordinary temperatures, was studied in acetone-water medium and in the absence of any catalyst. The pH of the reaction mixture was kept constant at 7.0 during the course of reaction.

The progress of reaction was followed spectrophotometerically on a Shimadzu double beam spectrophotometer, UV-150-02. However, stoichiometry was determined iodometrically. The reaction between PEA and periodate ion in acetone-water medium

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produced violet colour which showed maximum absorbance at 470 nm. The λ_{max} did not change during the period for which the kinetic studies were made.

RESULTS AND DISCUSSION

The reaction was studied at different concentrations of periodate and PEA respectively, keeping the other constant. Initial rates in terms of $(dA/dt)_i$ at different [PEA] and [Periodate], showed that the reaction follows second order kinetics, being first order each in PEA and periodate. The second order kinetics was also proved by the fact that the rate was linearly related to the concentration of the reactant varied in each case.

Under pseudo first order conditions, the plot of rate^{-1} vs $[S]^{-1}$ (where S is the reactant taken in excess) was linear with almost negligible intercept (Table-1), indicating that the intermediate formed in slow step got consumed in a subsequent fast step [1, 2]. The stoichiometry of the reaction determined by estimating the unreacted $NaIO_4$ iodometrically, was found to be 2:1 (oxidant-substrate).

The reaction mixture (PEA : $NaIO_4$=5:1) was filtered after 27 hours, filterate was extracted with petroleum ether and a major yellow colored component 'A' was collected by preparative tlc.

The melting point of component A was found to be 67.0^0 C. This compound was found to respond positively for a quinone[3,10].

The λ max obtained for this compound in C_2H_5OH solvent were foun to be 240 nm and 300 nm which suggests the presence of quinonoid structure in the compound [3,10].

The IR spectrum of the component A in KBr showed the presence of bands at 3023 cm^{-1} (S) (due to ring C-H strech), 2929 cm^{-1} (S) and 2871 cm^{-1} (S) (due to C-H streching vibrations in alkyl group), 1709 cm^{-1} (W) and 1895 cm^{-1} (W) (due to overtones and combination bands), 1674 cm^{-1} (S) (indicating the presence of quinonoid structure with the possibility of a +I effect of alkyl group causing a little lowering of position of this band [11, 12], 1516 cm^{-1} (S) and 1455 cm^{-1} (S) (due to C-------C ring strech). The bands were also obtained at 1169 cm^{-1} (m) to 1412 cm^{-1} (m) (may be due to the in plane C-H bending in the ring) and at 755 cm^{-1} (m) and 824 cm^{-1} (s) (due to out of plane C=C bending and C-H bending modes and substitution pattern in the ring).

The N.M.R. spectrum of this compound in $CDCl_3$ showed signals at δ =6.55,S, (IH); δ =7.01, D (IH); δ =7.32, D, (IH), δ =2.54, Q, (2H) and δ =1.2, T, (3H).

A triplet at δ =1.2 may be due to-CH_3 group protons while a quartet at δ =2.54 was due to-CH_2 group protons. The other signals in spectrum δ =6.55 to 7.32 were due to the three protons attached to the ring.

On the basis of these studies [10, 13], this compound may be 4-ethyl-1,2 benzoquinone.

Kinetic studies were carried out in the range of pH 4.0 to 7.6 using Thiel, Schultz and Coch Buffer [4] (Table-2). The rate increases sharply upto pH 4.48 to 6.24 which may be due to the decrease in protonation of PEA. The concentration of periodate monoanion is maximum around pH 5.0 to 7.0 and decreases beyond this pH value [1,2] which may probably be the reason for the decrease in rate beyond pH 6.24. After pH 6.24, the periodate dianion formation becomes predominant which is unreactive. A similar behaviour has been observed by previous workers in case of other anilines [1,5,6,9]

To get the further information about the participating reactants, different kinetic runs, under pseudo first order conditions were carried out in presence of different amounts of acetone ranging from 2.5% to 15.0% (v/v) (Table-3). On decreasing dielectric constant the rate was found decreasing. A plot between log $(dA/dt)_i$ vs I/D was found to be linear with negative slope indicating the reaction may be of ion-dipole type. The negative slope of this plot is in accordance with Amis view [7] that the slope will be negative if the reacting ion is anion, which is periodate monoanion in the present study.

To study the effect of varying ionic strength (μ) on the specific rate, the reaction was carried out under pseudo-first order conditions and in the presence of different concentrations of a neutral salt NaCl. The rate of reaction increased with an increase in ionic strength (Table-4) The plot between $(dA/dt)_i$ vs μ was of primary linear type, which indicates that the ion-dipole reaction is the rate determining step.

The kinetic studies were made under pseudo first order conditions (taking periodate in excess) at four different temperatures ranging from 30°C to 45°C. Guggenheim method was used for evaluating the first order rate constant and the second order rate constant

was calculated by dividing the first order rate constant by the concentration of periodate.

The linear arrhenius plot (log k_2 vs I/T) was used for calculating the thermodynamic parameters. The mean values of various activation parameters are, Ea=6.82 k. cal/mol; A=2.32×10^3 lit. mol^{-1} sec^{-1};

$\Delta S^{\#}$=-42.25 E.U.; $\Delta F^{\#}$=20.25 k.cal/mol and $\Delta H^{\#}$=6.20 k.cal/ mol.

From these data, it is clear that the reaction is characterized by a low value of energy of activation and a large negative value of entropy of activation. The former is the characteristic of a bimolecular reaction in solution and the latter is mainly observed in polar solvents and also suggests the formation of a charged and rigid transition state which is expected to be strongly solvated in the polar solvent employed. The above assumption is also supported by the fact that the rate decreases with decreasing dielectric constant. The value of frequency factor of the order of 10^3 is suggestive of the fact that the reactive species are large in size.

Before proposing a mechanism for this reaction, It is also to be noted that free radical scavengers have no effect on the rate of reaction. On the basis of the kinetic studies, insensitiveness towards free radical scavengers, product identified and the chances of the formation of benzoquinoneimine derivatives during such reactions as reported earlier [1,2] the proposed mechanism is given in chart-I.

The value of $\Delta S^{\#}$ and effect of dielectric constant suggests the formation of a charged intermediate (I) as shown in the mechanism. This intermediate (I) reacts with another molecule of periodate to form quinioneimine(II). The last step seems to be the fast hydrolysis of (II) to give (III) which was characterized by us as 4-ethyl-l, 2-benzoquinone.

On the basis of the above mechanism, the rate of the reaction should be given by

$$dA/dt = k_2[PEA][IO_4^-]$$

The mechanism proposed and the rate law derived is in accordance with various kinetic features observed, namely the second order kinetics, effect of pH, effect of dielectric constant, effect of ionic strength on rate, thermodynamic parameters evaluated and the product identified.

TABLE -1

EFFECT OF REACTANT CONCENTRATION ON RATE AT 30 ± 0.1°C

λ max = 470nm, Acetone=10.0 % (v/v)

[Substrate] $\times 10^3$M	[$NaIO_4$] $\times 10^3$M	[dA/dt]i $\times 10^2$
1.0	10.0	1.0
1.0	12.0	1.2
1.0	14.0	1.4
1.0	16.0	1.7
1.0	18.0	1.9
10.0	1.0	1.5
12.0	1.0	1.8
14.0	1.0	2.2
16.0	1.0	2.5
18.0	1.0	2.8

TABLE -2

EFFECT OF pH ON REACTION RATE

[P-EA]=0.001 M, [$NAIO_4$] = 0.01 M,

λ max = 470nm, Acetone=10.0 % (v/v)

Temp. = 30 ± 0.1°C

pH	4.48	5.12	5.48	6.24	6.54	7.08	7.56
$\left(\frac{dA}{dt}\right)_i \times 10^2$	1.30	1.60	1.85	2.20	1.80	1.40	1.15

TABLE -3

EFFECT OF DIELECTRIC CONSTANT

[P-EA]=0.001 M, $[NAIO_4]$ = 0.01 M,

λ max = 470nm, Temp. = 30 $\pm$ 0.1°C

D	73.9	72.4	70.0	66.8
$\left(\frac{dA}{dt}\right)_i \times 10^2$	2.0	1.5	1.0	0.55

TABLE -4

EFFECT OF IONIC STRENGTH

[P-EA]=0.001 M, $[NAIO_4]$ = 0.01 M,

λ max = 470nm, Acetone=10.0 % (v/v)

Temp. = 30 $\pm$ 0.1°C

[NaCl] x10^3M	$\mu \times 10^2$	$\left(\frac{dA}{dt}\right)_i \times 10^2$
1.0	1.1	1.15
3.0	1.3	1.50
5.0	1.5	2.00
7.0	1.7	2.80

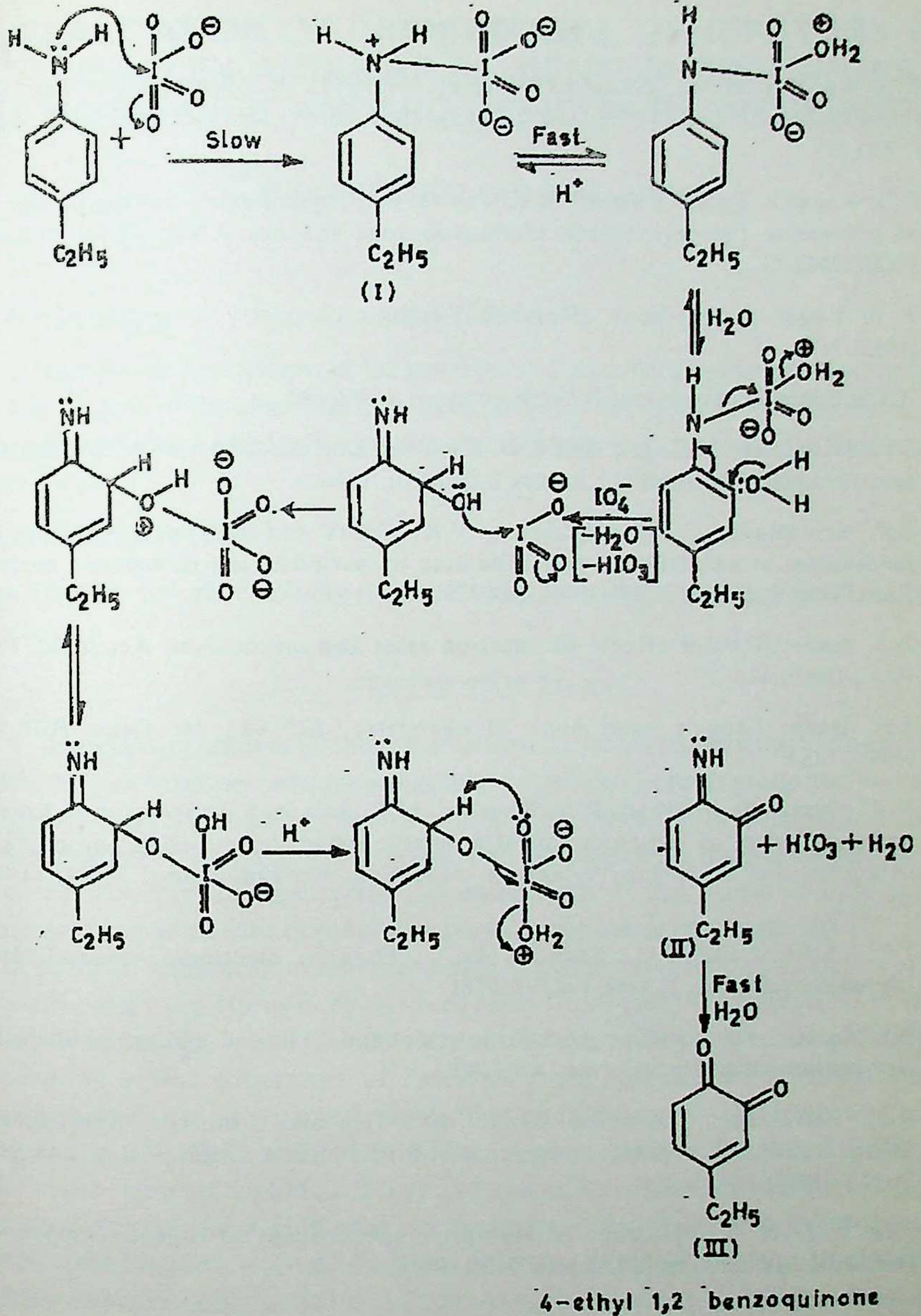

CHART
Oxidation of p-ethylaniline

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Journal of Natural & Physical Science, Vol. 14 (1-2) (2000) 79-96

REALIZATION OF INFINITESIMAL GENERATORS OF COMPLEX ANGULAR MOMENTUM OPERATORS IN TERMS OF STANDARD HELICITY REPRESENTATION

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(Received 29.07.2000)

ABSTRACT

Helicity representations of the generators of complex angular momentum operator have been derived for time-like, light-like and space-like cases and the results have been generalized to the standard helicity realization for all the mass systems, including imaginary mass.

PACS Number : 14.80.

Key Words : Standard helicity representation, space-like system, angular momentum operator.

INTRODUCTION

The classification of all the relativistic particles corresponding to the irreducible representation of proper orthochronous, inhomogeneous Lorentz group has been done by Wigner[1] who also found how the wave-function for these particles transform under the transformation of this group in the momentum representation. It has also been shown that how the reduction of unitary ray representations of this group and the reducible representation of the infinitesimal generators can be carried out explicitly [2, 3]. Moses derived the irreducible representations of the infinitesimal generators of this group for non-zero real mass (time-like) and zero mass (light-like) systems [4, 5] in terms of Foldy-Shirokov [6, 7] and Lomont-Moses [8] realizations respectively. Study of the realization of the generators of Poincare group, its unitary representation and transformation has been carried out by other workers also [9-11]. It was suggested by Wigner that only light-like and time-like representations have a direct application in relativistic elementary particle theory and space-like representations (imaginary mass systems) are unphysical. However, since last three decades a large number of papers, by distinguished theoretical physicists, on the different aspects of the imaginary mass particles (Tachyons) have been published [12-15]. As such, the space-like representations of the poincare group are making their appearance in several investigations with physical as well

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as mathematical point of views.

It is convenient for the classification of the unitary irreducible representation of the inhomogeneous Lorentz group to choose a definite basis in the vector space in which the representations are defined. To define a basis we select a complete set of commutating operators from among the infinitesimal operators of the group. We have used linear momentum and the standard helicity bases for developing the theory of space-like particles [16,17]. The generators of the complex angular momentum operators have been derived in terms of momentum representation in an earlier paper [18].

In the present paper the helicity representations of complex angular momentum operators have been derived for time-like, light-like and space-like cases and the results have been generalized to the standard helicity representation for all the three systems, including imaginary mass. It has been shown that all these representations have the forms similar to those derived by Shirokov [7] for the generators of ordinary angular momentum operator.

COMPLEX ANGULAR MOMENTUM OPERATORS IN STANDARD HELICITY REPRESENTATIONS:

Discrete-spin light-like case gives one-dimensional representations characterized by the eigenvalues of the helicity operator

$$\hat{\lambda} = \frac{\vec{J}.\vec{P}}{\left|\vec{P}\right|} = \frac{\vec{S}.\vec{P}}{\left|\vec{P}\right|} \tag{1}$$

In all the cases discussed in earlier [18], the Chakrabarti's [10] transformations

$$U_1 = \exp\left[i\theta \frac{p_1 S_2 - p_2 S_1}{\left(p_1^2 + p_2^2\right)}\right] \tag{2a}$$

may be written, as a real rotation in spin-space with Euler angles $(\phi, -\theta, -\phi)$, in the following form

$$U_1 = \exp(-i\phi S_3)\,\exp(i\,\theta S_2)\exp(i\,\phi S_3) \tag{2b}$$

where the polar angles θ and ϕ of $\vec{p}$ in an initial frame are given by

$$\theta = arc\ tg\ \frac{\left(p_1^2 + p_2^2\right)^{1/2}}{p_3} = \cos^{-1}\left(\frac{p_3}{p}\right) \tag{3a}$$

and $$\phi = \tan^{-1}\left(\frac{p_2}{p_1}\right) \tag{3b}$$

It can be shown that equation (2b) diagonalizes the helicity operator given by equation (1) to usual diagonal spin operator S_3 as follows;

$$U_1 \hat{\lambda} U_1^{-1} = S_3 \tag{4}$$

Here we have used three irreducible matrices (S_1, S_2, S_3) of (2s+1) dimensions of arbitrary spin-s, such that

$$\left[S_j, S_k\right] = i\varepsilon_{jkn}\ S_n,\ (j,k,n = 1,2,3) \tag{5}$$

where ε_{ijk} is usual Levi- Cevita three index symbols.

The property, given by equation (4), plays an important role in the definition of helicity basis and standard helicity representations of the generators of complex angular momentum operators. In the following sections we have discussed these representations for real non-zero, zero and imaginary mass systems separately.

REAL NON-ZERO MASS (TIME-LIKE) CASE:

Transformation of the complex angular momentum operator for non-zero mass case under the Chakrabarti's unitary operator U_1 given by equation (2) may also be written as,

$$\hat{Z}_T = -\frac{i}{2}\vec{p}\times\frac{\partial}{\partial \vec{p}} + \frac{1}{2}S_3\vec{X} - \frac{1}{2}w\frac{\partial}{\partial \vec{p}} - \frac{iwS_3}{2p}\left(\vec{e}\times\vec{X}\right) + \frac{im}{2p}\vec{\alpha}\ , \tag{6a}$$

where

$$\vec{\alpha} = \vec{S}\times\vec{e} + \frac{S_1p_2 - S_2p_1}{p(p+p_3)}\left(\vec{p} + p\vec{e}_3\right), \tag{6b}$$

and

$$\vec{X} = \begin{bmatrix} \dfrac{p_1}{p+p_3} \\ \dfrac{p_2}{p+p_3} \\ 1 \end{bmatrix}, \quad \vec{e} = (0,0,1) \tag{7}$$

The components of complex angular momentum operator have been defined in the following form [19-21];

$$\hat{Z}_j = \frac{1}{2}\left(\hat{J}_j + i\hat{K}_j\right) \tag{8}$$

where $\hat{J}_j$ are the generators corresponding to the ordinary angular momentum operators associated with rotation and $\hat{K}_j$ are the generators for pure Lorentz transformations. It can be easily shown that

$$\left(\vec{S}.\vec{p}\right) = \left(S_3\,\vec{X}\right).\,\vec{p} = pS_3\ , \tag{9}$$

which gives the condition (4) for the diagonalization of helicity operator.

Let us consider the canonical basis as the set of the state vectors $\left|[m,s],\vec{p},\sigma\right\rangle$ which describe the states of mass m, spin-s, momentum $\vec{p}$ and the z- component of angular momentum σ. The transformation of these state vectors under homogeneous Lorentz transformations is well known.

It has already been discussed in an earlier paper [19] that the generators corresponding to complex angular momentum operators are associated with the complex rotations as follows ;

$$\vec{\phi} = 2\left(\vec{\theta} + i\vec{\beta}\right) \tag{10}$$

i.e. real rotation followed by real Lorentz transformation. We may therefore, write the transformation of the state vectors in the helicity basis under the operators of $\hat{Z}_T$ given by equation (6) in the following form,

$$\psi\left(\vec{p}+\vec{\phi}\times\vec{p}\right)=\left\{1-i\vec{\phi}\left[\frac{1}{2}S_3\vec{X}-\frac{iw}{2p}S_3\vec{e}_3\times\vec{X}+\frac{im}{2p}\vec{\alpha}\right]\right\}\Psi\left(\vec{p}\right), \tag{11}$$

which may also be written as

$$\psi\left(\vec{p}+\vec{\phi}\times\vec{p}\right)=\left[1-i\left(\vec{\theta}.\vec{a}+\vec{\beta}.\vec{b}\right)\right]\Psi\left(\vec{p}\right) \tag{12}$$

where $\vec{a}=S_3\vec{X}$ *and* $\vec{b}=\dfrac{w}{p}S_3\vec{e}_3\times\vec{X}-\dfrac{m}{p}\vec{\alpha}$ (13)

Under the Chakrabarti's unitary transformation U_1 for non-zero real mass case the helicity realization of the generators associated with complex angular momentum operator may be written in the following manner

$$\hat{Z}_{T_1}=\frac{1}{2}\left[\begin{array}{c}-i\left(\vec{p}\times\vec{\nabla}\right)_1+\dfrac{p_1S_3}{p+p_3}-\varepsilon w\vec{\nabla}_1+i\varepsilon w\dfrac{p_2S_3}{p(p+p_3)}\\ -\dfrac{i\varepsilon w}{p^2}\dfrac{p_1}{p+p_3}\left(\vec{p}\times\vec{S}\right)_3+\dfrac{i\varepsilon m}{p}S_2\end{array}\right]$$

$$\hat{Z}_{T_2}=\frac{1}{2}\left[\begin{array}{c}-i\left(\vec{p}\times\vec{\nabla}\right)_2+\dfrac{p_2S_3}{p+p_3}-\varepsilon w\vec{\nabla}_2+i\varepsilon w\dfrac{p_1S_3}{p(p+p_3)}\\ -\dfrac{i\varepsilon w}{p^2}\dfrac{p_2}{p+p_3}\left(\vec{p}\times\vec{S}\right)_3+\dfrac{i\varepsilon m}{p}S_1\end{array}\right]$$

$$\hat{Z}_{T3}=\frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_3-\varepsilon w\vec{\nabla}_3+\frac{i\varepsilon m}{p^2}\left(\vec{p}\times\vec{S}\right)_3+S_3\right] \tag{14}$$

Where $\vec{\nabla}=\dfrac{\partial}{\partial\vec{p}}$, $w=\left(m^2+p^2\right)^{1/2}$ and ε is the sign of energy i.e. $\varepsilon=\pm1$.

Let us now redefine the following generators of the little group for real non- zero mass system;

$$\hat{T}_1 = -\hat{S}_2 \,,$$

$$\hat{T}_2 = -\hat{S}_1 \,, \tag{15}$$

$$\hat{M} = \hat{S}_3 \,,$$

such that they satisfy the commutation relations,

$$\left[\hat{T}_1, \hat{M}\right] = -i\,\hat{T}_2 \quad ,$$

$$\left[\hat{T}_2, \hat{M}\right] = i\,\hat{T}_1 \quad ,$$

$$\left[\hat{T}_1, \hat{T}_2\right] = i\,\hat{M} \quad , \tag{16}$$

Then the helicity representation (14) of the complex angular momentum operators may be written in the following form,

$$\hat{Z}_{T_1} f\left(m,\varepsilon,\vec{p},\lambda\right) = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_1 + \frac{p_1\hat{M}}{p+p_3} - \varepsilon w\nabla_1 + i\varepsilon w\frac{p_2\hat{M}}{p(p+p_3)} + i\varepsilon\frac{m}{p^2}\frac{p_1}{(p+p_3)}\left(\vec{p}.\vec{T}\right) - \frac{i\varepsilon m}{p}\hat{T}_1\right] f\left(m,\varepsilon,\vec{p},\lambda\right),$$

$$\hat{Z}_{r_2} f\left(m,\varepsilon,\vec{p},\lambda\right) = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_2 + \frac{p_2\hat{M}}{p+p_3} - \varepsilon w\nabla_2 + i\varepsilon w\frac{p_1\hat{M}}{p(p+p_3)} + i\varepsilon\frac{m}{p^2}\frac{p_1}{(p+p_3)}\left(\vec{p}.\vec{T}\right) - \frac{i\varepsilon m}{p}\hat{T}_2\right] f\left(m,\varepsilon,\hat{p},\lambda\right),$$

$$\hat{Z}_{r_3} f\left(m,\varepsilon,\vec{p},\lambda\right) = \frac{1}{2}\left[-i\left(\vec{p}.\vec{\nabla}\right)_3 - \varepsilon w\nabla_3 + \frac{i\varepsilon m}{p}\left(\vec{p}.\vec{T}\right) + \hat{M}\right] f\left(m,\varepsilon,\vec{p},\hat{\lambda}\right)$$

where $\quad \vec{p}.\vec{T} = p_1T_2 + p_2T_1$ (18)

and the differential complex function $f(m,\varepsilon,\vec{p},\lambda)$ represents every space vector in the representation space, λ denotes the eigenvalues of the helicity operator defined by (1), ε is the sign of energy, $\vec{p}$ denotes collectively three real variables p_1, p_2, p_3, each of which can take on all real values and the variable m takes on the eigenvalues of the mass operator given as follows;

$$\hat{C} = \hat{H}^2 - \sum_{j-1}^{3} \hat{p}^2 , \tag{19}$$

where $\hat{H}$ is the Hamiltonian operator and P_j are operators corresponding to three components of linear momentum. The operator $\hat{C}$, known as Casimir operator, commutes with the infinitesimal generators of the homogeneous Lorentz group and its eigenvalues c are given by the following equations for the three cases i.e. time-like, light-like and space-like;

$$\begin{aligned} c &= m^2 && \text{for real non-zero mass case,} \\ c &= 0 && \text{for zero mass case,} \\ c &= -\mu^2 && \text{for imaginary mass case.} \end{aligned} \tag{20}$$

We may write the standard helicity realizations in terms of the infinitesimal generators acting on the function $f(c,\varepsilon,\vec{p},\lambda)$ as follows;

$$\hat{p}_j f(c,\varepsilon,\vec{p},\lambda) = p_j \, f(c,\varepsilon,\vec{p},\hat{\lambda})$$

$$\hat{H} f(c,\varepsilon,\vec{p},\hat{\lambda}) = \varepsilon \, f(c,\varepsilon,\vec{p},\hat{\lambda}) \tag{21}$$

Equation (17) leads to the so called standard helicity realization derived by Moses [4] From this realization we may readily derived the following Foldy-Shirokov [6,7,22] realization;

$$\hat{Z}_r = \left[-i\left(\vec{p} \times \vec{\nabla}\right) + \vec{S} \right] g\left(\vec{p},\lambda\right), \tag{22}$$

where $g\left(\vec{p},\lambda\right) = \left[\exp\left(i\,\vec{\theta}.\vec{S} \right) \exp\left(i\frac{\pi}{2} S_3 \right) \right] f(c,\varepsilon,\vec{p},\lambda)$

and $\theta = \theta_1^2 + \theta_2^2 = \cos^{-1}\frac{p_3}{p}$

$$\theta_1 = \frac{p_2}{\left(p_1^2 + p_2^2\right)} \cos^{-1}\frac{p_3}{p}$$

$$\theta_2 = \frac{-p_1}{\left(p_1^2 + p_2^2\right)} \cos^{-1}\frac{p_3}{p}$$

ZERO-MASS SYSTEM (LIGHT-LIKE CASE) :

We have already seen that the representations of the complex angular momentum operator for discrete-spin light-like case follows from those for real non-zero mass case in the limit $m \to 0$. As such analogue of Equation (7) may be written as

$$\hat{Z}_{LD} = -\frac{i}{2}\vec{p}\times\frac{\partial}{\partial\vec{p}} + \frac{1}{2}S_3\vec{X} - \frac{1}{2}p\frac{\partial}{\partial\vec{p}} - \frac{i}{2}S_3\left(\vec{e}_3\times\vec{X}\right) \tag{23}$$

Similarly equation (12) reduces to the following form in the limit $m \to 0$;

$$\Psi\left(\vec{p}+\vec{\phi}\times\vec{p}\right) = \left[1 - i\lambda\left\{\vec{\theta}.\vec{X} + \vec{\beta}\times\left(\vec{e}_3\times\vec{X}\right)\right\}\right]\Psi\left(\vec{p}\right) \tag{24}$$

which is identical to Fonda's [23] result for unitary component representation of zero-mass case. For light-like discrete-spin case ($T_1=T_2=0$), the helicity realization for the generators of complex angular momentum operator may also be written as follows;

$$\hat{Z}_{LD_1} f\left(0,\varepsilon,\vec{p},\lambda\right) = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_1 + \frac{p_1\hat{M}}{p+p_3} - \varepsilon p\nabla_1 + \frac{i\varepsilon p_2\hat{M}}{p+p_3}\right] f(0,\varepsilon,\vec{p},\lambda)$$

$$\hat{Z}_{LD_2} f\left(0,\varepsilon,\vec{p},\lambda\right) = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_2 + \frac{p_2\hat{M}}{p+p_3} - \varepsilon p\nabla_2 + \frac{i\varepsilon p_1\hat{M}}{p+p_3}\right] f(0,\varepsilon,\vec{p},\lambda)$$

$$\hat{Z}_{LD_3} f\left(0,\varepsilon,\vec{p},\lambda\right) = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_3 - \varepsilon p\nabla_3 + \hat{M}\right] f(0,\varepsilon,\vec{p},\lambda) \tag{25}$$

where $\hat{M} = S_3$. These standard helicity realizations for discrete-spin light like case may also be written directly from (17) in the limit $m \to 0$ and defining $\widetilde{M} = S_3$ such that

$$\frac{p_1}{p+p_3}\widetilde{M} = \widetilde{S}_1, \frac{p_1}{p+p_3}\widetilde{M} = \widetilde{S}_2, \widetilde{\nabla} = \frac{1}{2}\left(\vec{\nabla} - \frac{\varepsilon\vec{S}}{p}\right), \tag{26}$$

Therefore, we may write equation (25) in the following form

$$\hat{Z}_{LD_1} f\left(0,\varepsilon,\vec{p},\lambda\right) = \left[-i\left(\vec{p}\times\vec{\nabla}\right)_1 + \frac{p_1\widetilde{M}}{p+p_3}\right] f(0,\varepsilon,\vec{p},\lambda)$$

$$\hat{Z}_{LD_2} f\left(0,\varepsilon,\vec{p},\lambda\right) = \left[-i\left(\vec{p}\times\vec{\nabla}\right)_2 + \frac{p_2\widetilde{M}}{p+p_3}\right] f(0,\varepsilon,\vec{p},\lambda)$$

$$\hat{Z}_{LD_3} f\left(0,\varepsilon,\vec{p},\lambda\right) = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_3 + \hat{M}\right] f(0,\varepsilon,\vec{p},\lambda) \tag{27}$$

which give the realization of complex angular momentum operators for discrete-spin mass less case in the form similar to those for ordinary angular momentum operator in standard helicity representation except that the operators $\vec{\nabla}$ and $\vec{M}$ are replaced by the operators $\widetilde{\nabla}$ and $\widetilde{M}$ respectively.

For one valued representation (integer eigenvalues of helicity operator) in continuous-spin mass less case the helicity realization of complex angular momentum operator be written in the following form;

$$\hat{Z}^{(a)}_{LC_1} = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_1 + \frac{p_1 S_3}{p+p_3} - \frac{i\varepsilon p_2}{p+p_3}S_3 - \varepsilon p\nabla_1\right.$$

$$\left. + \frac{i\varepsilon p}{p^2(p+p_3)}\left(\vec{p}\times\vec{T}\right)_3 - \frac{i\varepsilon\hat{T}_2}{p} -\right]$$

$$\hat{Z}^{(a)}_{LC_2} = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_2 + \frac{p_2 S_3}{p+p_3} - \frac{i\varepsilon p_1}{p+p_3}S_3 - \varepsilon p\nabla_2\right.$$

$$+\frac{i\varepsilon p_2}{p^2(p+p_3)}\left(\vec{p}\times\vec{T}\right)_3-\frac{i\varepsilon\hat{T}_1}{p}-\Bigg]$$

$$\hat{Z}^{(a)}_{LC\,3} = \frac{1}{2}\Bigg[-i\left(\vec{p}\times\vec{\nabla}\right)_3+S_3-\varepsilon p\nabla_3+\frac{i\varepsilon}{p^2}\left(\vec{P}\times\vec{T}\right)_3 \tag{28}$$

Let us redefine the generators of little group in the following form;

$$\begin{aligned}\hat{T}_1 &= \hat{T}_2 \ ,\\ \hat{T}_2 &= -\hat{T}_1 \ ,\\ \hat{M} &= \hat{S}_3 \ ,\end{aligned} \tag{29}$$

Such that they satisfy the commutation relations;

$$\begin{aligned}\left[\hat{T}_1 \ , \ \hat{T}_2\right] &= 0,\\ \left[\hat{T}_1 \ , \ \hat{M}\right] &= -i\hat{T}_2,\\ \left[\hat{T}_2 \ , \ \hat{M}\right] &= i\hat{T}_1,\end{aligned} \tag{30}$$

Which are the usual commutation rules for the generators of the Euclidean group E(2). These new generators may also be written as;

$$\hat{M}=\hat{J}_1,\ \ \hat{T}_1=\hat{J}_1+\hat{K}_2,\ \hat{T}_2=\hat{J}_2+\hat{K}_1 \tag{31}$$

Substituting these values of the generators in equation (28), we get the following standard helicity realization of complex angular momentum operator for single valued representation in continuous spin mass less case;

$$\hat{Z}^{(a)}_{LC_1}f(0,\varepsilon,\vec{p},\lambda)=\frac{1}{2}\Bigg[-i\left(\vec{p}\times\vec{\nabla}\right)_1+\frac{p_1\hat{M}}{p+p_3}-\frac{i\varepsilon p_2}{p+p_3}\hat{M}-\varepsilon p\nabla_1$$

$$+\frac{i\varepsilon p}{p^2(p+p_3)}\left(\vec{p}.\vec{T}\right)-\frac{i\varepsilon\hat{T}_1}{p}\Bigg]f(0,\varepsilon,\vec{p},\lambda)$$

$$\hat{Z}^{(a)}_{LC_2} f(0,\varepsilon,\vec{p},\lambda) = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_1 + \frac{p_2\hat{M}}{p+p_3} - \frac{i\varepsilon p_1}{p+p_3}\hat{M} - \varepsilon p\nabla_2 \right. \tag{32}$$

$$\left. + \frac{i\varepsilon p}{p^2(p+p_3)}\left(\vec{p}.\vec{T}\right) - \frac{i\varepsilon\hat{T}_2}{p}\right] f(0,\varepsilon,\vec{p},\lambda)$$

$$\hat{Z}^{(a)}_{LC_3} f(0,\varepsilon,\vec{p},\lambda) = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_3 + \hat{M} - \varepsilon p\nabla_3 + \frac{i\varepsilon}{p^2}\left(\vec{p}.\vec{T}\right)\right] f(0,\varepsilon,\vec{p},\lambda)$$

Where $\vec{p}.\vec{T} = p_1\hat{T}_1 + p_2\hat{T}_2$ These equations lead to Lomont-Moses realization [8] and standard helicity realization derived by Moses 8 for zero mass system.

For two-valued representation (half integer eigen-values of helicity operator) in continuous-spin mass-less case the helicity realization of complex angular momentum operator may be written to the following form;

$$\hat{Z}^{(a)}_{LC_1} = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_1 + \frac{p_1 S_3}{p+p_3} - \frac{i\varepsilon p_2}{p+p_3} S_3 - \varepsilon p\nabla_1 \right.$$

$$\left. + \frac{i\varepsilon p}{p^2(p+p_3)}\left(\vec{p}\times\vec{T}\right)_3 - \frac{i\varepsilon\hat{T}_2}{p}\right]$$

$$\hat{Z}^{(a)}_{LC_2} = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_2 + \frac{p_2 S_3}{p+p_3} - \frac{i\varepsilon p_1}{p+p_3} S_3 - \varepsilon p\nabla_2 \right.$$

$$\left. + \frac{i\varepsilon p_2}{p^2(p+p_3)}\left(\vec{p}\times\vec{T}\right)_3 - \frac{i\varepsilon\hat{T}_1}{p} - \right]$$

$$\hat{Z}^{(a)}_{LC_3} = \frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_3 + S_3 - \varepsilon p\nabla_3 + \frac{i\varepsilon}{p^2}\left(\vec{p}\times\vec{T}\right)_3\right] \tag{33}$$

which lead to the standard helicity realization (32) if we redefine the generators of the little group as follows;

$$\hat{T}_1 = -\hat{T}_2 = \hat{J}_2 - \hat{K}_1,$$

$$\hat{T}_2 = -\hat{T}_1 = -\hat{J}_1 - \hat{K}_2, \tag{34}$$

which satisfy the usual commutation rules (30) of the generators of Euclidean group E (2) Let us now consider a special case of Equ. (32) by restricting the generators

$\hat{T}_1$, $\hat{T}_2$ and $\hat{M}$ to satisfy the following equations;

$$\hat{T}_1 = \frac{p_1}{p+p_3}\left(\vec{p}\times\vec{S}\right)_3 ,$$

$$\hat{T}_2 = \frac{p_2}{p+p_3}\left(\vec{p}\times\vec{S}\right)_3 , \tag{35}$$

which satisfy commutation relations (30) only in the restricted case $\vec{p} = (0,0,p_3)$ Then (32) reduces to

$$\hat{Z}_{LC1} f(0,\varepsilon,\vec{p},\lambda) = \left[-i\left(\vec{p}\times\tilde{\nabla}\right)_1 + \frac{1}{2}\frac{p_1}{p+p_3}\hat{M} - \frac{1}{2}\varepsilon p\frac{\partial}{\partial p_1}\right] f(0,\varepsilon,\vec{p},\lambda)$$

$$\hat{Z}_{LC2} f(0,\varepsilon,\vec{p},\lambda) = \left[-i\left(\vec{p}\times\tilde{\nabla}\right)_2 + \frac{1}{2}\frac{p_1}{p+p_3}\hat{M} - \frac{1}{2}\varepsilon p\frac{\partial}{\partial p_2}\right] f(0,\varepsilon,\vec{p},\lambda)$$

$$\hat{Z}_{LC3} f(0,\varepsilon,\vec{p},\lambda) = \left[-i\left(\vec{p}\times\tilde{\nabla}\right)_3 + \frac{1}{2}\frac{p_1}{p+p_3}\hat{M} - \frac{1}{2}\varepsilon p\frac{\partial}{\partial p_3}\right] f(0,\varepsilon,\vec{p},\lambda) \tag{36}$$

where $\vec{\nabla}$ has been defined by (26). These equations reduce to (27) for $\hat{M}$ defined by (26) and hence for continuous-spin mass less case also the standard helicity realization of generators of complex angular momentum operator may be written in the form similar to that for the generators of ordinary angular momentum operator except that $\hat{M}$ is replaced by $\hat{M}$ and $\vec{\nabla}$ by $\tilde{\nabla}$.

IMAGINARY MASS (SPACE-LIKE) CASE:

For space-like case the Wigner [24] little group is the three dimensional homogeneous Lorentz group SO(2,1) which is homomorphic to the spinor group SU(1,1). Its generators

$$\hat{J}_3;\hat{R}_1=-\hat{K}_2;\hat{R}_2=\hat{K}_1 \tag{37}$$

satisfy the following commutation relations;

$$\left[\hat{J}_3,\hat{R}_r\right]=i\varepsilon_{rs}\vec{R}_s$$

$$\left[\hat{R}_r,\hat{R}_s\right]=i\varepsilon_{rs}J_3 \tag{38}$$

r,s= 1,2 and ε_{rs} is two-dimensional antisymmetric matrix with $\varepsilon_{12}=-\varepsilon_{21}=-1$.

The realization of the generators associated with complex angular momentum operator for imaginary mss system may also be written as follows, in the similar fashion as discussed above for real non-zero and zero mass cases;

$$\hat{Z}_{s_1}=\frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_1+\frac{p_1}{p+p_3}S_3-\varepsilon w\nabla_1+\frac{i\varepsilon wp_2}{p(p+p_3)}+\frac{i\varepsilon\mu}{p^2}\frac{p_1}{p+p_3}\left(\vec{p}\times\vec{R}\right)_3-\frac{i\varepsilon\mu}{p}\hat{R}_2\right]$$

$$\hat{Z}_{s_2}=\frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_2+\frac{p_2}{p+p_3}S_1-\varepsilon w\nabla_2+\frac{i\varepsilon wp_1S_3}{p(p+p_3)}+\frac{i\varepsilon\mu}{p^2}\frac{p_2}{p+p_3}\left(\vec{p}\times\vec{R}\right)_3-\frac{i\varepsilon\mu}{p}\hat{R}_1\right]$$

$$\hat{Z}_{s_3}=\frac{1}{2}\left[-i\left(\vec{p}\times\vec{\nabla}\right)_3+S_3-\varepsilon w\nabla_3+\frac{i\varepsilon\mu}{p^2}\left(\vec{p}\times\vec{R}\right)_3\right] \tag{39}$$

where $m=i\mu$, which is imaginary.

Let us now redefine the generators of little group for imaginary mass case as follows;

$$\hat{T}_1'=\hat{R}_2=\hat{K}_1,$$

$$\hat{T}_2'=-\hat{R}_1=\hat{K}_2, \tag{40}$$

$$\hat{M} = \hat{S}_3 = \hat{J}_3,$$

such that they satisfy the commutation relations,

$$\left[\hat{T}_1, \hat{M}\right] = -i\hat{T}_2,$$

$$\left[\hat{T}_2, \hat{M}\right] = +i\hat{T}_1, \tag{41}$$

$$\left[\hat{T}_1, \hat{T}_2\right] = -i\hat{M},$$

Then the helicity realization of complex angular momentum operator given by equation (39) may be written in the following form;

$$\hat{Z}_{s_1} f(\mu, \varepsilon, \bar{p}, \lambda) = \frac{1}{2}\left[-i\left(\bar{p} \times \vec{\nabla}\right)_1 + \frac{p_1 \hat{M}}{p + p_3} - \varepsilon w \nabla_1 + i\varepsilon w \frac{p_2 \hat{M}}{p(p + p_3)}\right.$$

$$\left. + i\varepsilon \frac{\mu}{p^2} \frac{p_1}{(p + p_3)}\left(\bar{p}.\vec{T}''\right) - \frac{i\varepsilon\mu}{p} T_1''\right] f(\mu, \varepsilon, \bar{p}, \lambda)$$

$$\hat{Z}_{s_2} f(\mu, \varepsilon, \bar{p}, \lambda) = \frac{1}{2}\left[-i\left(\bar{p} \times \vec{\nabla}\right)_2 + \frac{p_2 \hat{M}}{p + p_3} - \varepsilon w \nabla_2 + i\varepsilon w \frac{p_1 \hat{M}}{p(p + p_3)}\right.$$

$$\left. + i\varepsilon \frac{\mu}{p^2} \frac{p_2}{(p + p_3)}\left(\bar{p}.\vec{T}''\right) - \frac{i\varepsilon\mu}{p} T_2''\right] f(\mu, \varepsilon, \bar{p}, \lambda)$$

$$\hat{Z}_{s_3} f(\mu, \varepsilon, \bar{p}, \lambda) = \frac{1}{2}\left[-i\left(\bar{p} \times \vec{\nabla}\right)_3 + \hat{M} - \varepsilon w \nabla_3\right.$$

$$\left. + i\varepsilon \frac{\mu}{p^2}\left(\bar{p}.\vec{T}''\right)\right] f(\mu, \varepsilon, \bar{p}, \lambda) \tag{42}$$

where $\bar{p}.\vec{T}'' = p_1\hat{T}_1 + p_2 T_2''$

For two valued representation in continuous imaginary mass case, the helicity realization of complex angular momentum operator may also be written in the form given by equation (40) defining the generators of little group as,

$$\hat{T}_1'' = -\hat{R}_1 = \hat{K}_2$$

$$\hat{T}_2'' = -\hat{R}_2 = -\hat{K}_1 \tag{43}$$

The commutation rules (16), (30) and (41) for all the three cases may be generalized in the following form,

$$\left[\hat{T}_1 , \hat{T}_2\right] = i\frac{c}{|c|}\hat{M},$$

$$\left[\hat{T}_1'', \hat{M}\right] = -i\hat{T}_2, \tag{44}$$

$$\left[\hat{T}_2'', \hat{M}\right] = i\hat{T}_1,$$

where c is the eigenvalue of the Casimir operator $\hat{C}$ given by (19) and (20).

From eqs. (17), (32) and (42), the generators of the complex angular momentum operator for non-zero real mass, zero- mass and imaginary mass cases with continuous spins may be written in standard helicity realization as follows;

$$\hat{Z}_1 f(c,\varepsilon,\bar{p},\lambda) = \frac{1}{2}\Big[-i\left(\bar{p}\times\vec{\nabla}\right)_1 + \frac{p_1\hat{M}}{p+p_3} - \varepsilon w(c,p)\nabla_1 + \frac{i\varepsilon w(c,p)p_2\hat{M}}{p(p+p_3)}$$

$$+\frac{i\varepsilon B(c)}{p^2}\frac{p_1}{(p+p_3)}\left(\bar{p}.\bar{T}\right) - \frac{i\varepsilon B(c)}{p}\hat{T}_1\Big]\ f(c,\varepsilon,\bar{p},\lambda),$$

$$\hat{Z}_2 f(c,\varepsilon,\bar{p},\lambda) = \frac{1}{2}\Big[-i\left(\bar{p}\times\vec{\nabla}\right)_2 + \frac{p_2\hat{M}}{p+p_3} - \varepsilon w(c,p)\nabla_2 - \frac{i\varepsilon w(c,p)p_1\hat{M}}{p(p+p_3)}$$

$$+\frac{i\varepsilon B(c)}{p^2}\frac{p_2}{(p+p_3)}\left(\bar{p}.\bar{T}\right) - \frac{i\varepsilon B(c)}{p}\hat{T}_2\Big]\ f(c,\varepsilon,\bar{p},\lambda),$$

$$\hat{Z}_3 f(c,\varepsilon,\vec{p},\lambda) = \frac{1}{2}\Big[-i\big(\vec{p}\times\vec{\nabla}\big)_3 - \varepsilon w(c,p)\nabla_3 + \hat{M}$$

$$+\frac{i\varepsilon B(c)}{p^2}\big(\vec{p}.\vec{T}\big)\Big]\ f(c,\varepsilon,\vec{p},\lambda), \tag{45}$$

where $B(c) = \big[|c|\big]^{1/2}$ $\quad for\ \ c \neq 0,\ \ and\ \ w(c,p) = \ (p^2 + c)1/2$ (46)

Eqs. (45) lead to the standard helicity realization derived by Moses[4]. Standard helicity realization given in (27) for discrete-spin case readily follows from (15) by putting c=0 and $T_1 = T_2 = 0$. In other words, the determination of the standard helicity realization of the continuous-spin mass-less case automatically gives the corresponding realization for discrete-spin case, as a special case.

DISCUSSION

We have obtained the standard helicity realization of infinitesimal generators of complex angular momentum operator for time-like, light-like and space-like cases. In order to obtain these realizations we have used the property of the Chakrabarti's transformation matrix in the form given by equation (2b) which diagonalizes the helicity operator given by (1). For the time-like case the transformation of state vectors in the helicity basis under the operations of complex angular momentum operators has been derived in the form of equation (11) and the standard helicity realization of the generators of complex angular momentum operator has been obtained as in equation (17) by redefining the generators of the little group SO (3) in the form of equation (15). This realization leads to the usual standard helicity realization (21) given by Moses[4] and Foldy-Shirokov [6,7,22]. For the discrete-spin light-like case, the transformation of state vector in helicity basis has been obtained in the form of equation (24) which is identical to Fronsdal's [23] result for unitary one component representation of zero-mass case. The standard helicity realization of the generators of complex angular momentum operator in this case has been obtained in equation (25) which have the form similar to that for ordinary angular momentum operator in standard helicity representation[4] except that the operator $\vec{\nabla}$ is replaced by $\tilde{\nabla}$ and the operator $\hat{M}$ is replaced by $\tilde{M}$. The standard helicity representation given by equation (32) for one valued representation (integer eigenvalues of helicity operator) in continuous spin light-like case leads to Lomont Moses realization[8] and the standard helicity realization derived by Moses[4] for zero-mass system. Standard helicity representation given by equation (33) for the generators of complex angular momentum operator for two-valued representation

(half-integer eigenvalues of helicity operator) in continuous-spin light-like case, also reduces to that given by equations (32) when the generators of the Euclidean group E (2) are defined in the form given by equations (36) which give equation (25). Thus, for continuous-spin mass less case also the standard helicity realizations of complex angular momentum operator may be written in the form similar to that for the generators of ordinary angular momentum operator.

For one-valued and two-valued representations in continuous-spin space-like case the standard helicity representations of the complex angular mometum operators have been obtained in the form given by equation (42) by redefining the generators of the little group SO (2,1) in the form given by equation (40) and (43) respectively. The standard helicity realization of time-like, light-like and space-like cases have been generalized in the form of equations (45) and it has been shown that the determination of standard helicity realization of the continuous-spin mass less case automatically gives the corresponding realization for discrete-spin as a special case. It may be concluded, from the various realizations discussed above, that the specific discrete and continuous-spin realizations of mass less case are expressible in terms of Foldy-Shirokov forms while for space-like case such connection exists only for the representations of the little group SO (2,1) or the homomorphic spin group SU(1,1) belonging to the continuous series.

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Journal of Natural & Physical Sciences, Vol. 14 (1-2) (2000) 97-104

A PSEUDO DYNAMIC THRESHOLD SCHEME#

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(Received 04.02.2000 and after revision 29.08.2000)

ABSTRACT :

The present paper redescribes " Dynamic threshold schemes" Proposed by CHI-SUNG LAIH et al [6] as a "pseudo dynamic threshold scheme" and also gives a geometrical construction. In a **(d,m,n,T) dynamic threshold scheme,** there are n secret shadows (shares) and a public shadow p^j, at time $t=t_j$ (j varying from 1 to T). After knowing any m shadows, and the public shadow p^j, one can easily recover d master keys, K^j_i, $i=1,....,d$. Furthermore, if d master keys have to be changed to K_i^{j+1}, $i=1,....,d.$ for some security reasons, only the public shadow p^j has to be changed to p^{j+1}. Though, according to CHI-SUNG LAIH et al, it is very difficult to design an ideal dynamic threshold scheme to satisfy the required three characteristics, but after combining geometry and discrete log with some assumptions one can get a scheme whose function is same as that of dynamic threshold scheme.

Mathematical Subject Classification Number : 92 D : 94O21, 94O34, 94A60
Keywords and Phrases: Secret, Dynamic threshold scheme, Discrete log, Share, Shadow, Perfect security, Key distribution center, One way function.

INTRODUCTION

A (m,n)- threshold scheme is a method of distributing secret among n participants such that:

(i) any m (or more) out of n participants can reconstruct the secret, and

(ii) Fewer than m participants can learn nothing about the secret.

In a **(d,m,n,T) dynamic threshold scheme,** there are n secret shadows (shares) and a public shadow p^j, at time $t = t_j, 1 \le t_j \le T$. After knowing any m shadows, $m \le n$, and the public shadow p^j, we can easily recover d master keys, $K^j_i, i=1,....,d$. Furthermore, if d master keys have to be changed to $K^{j+1}_i, i= 1,....,d.$ for some security reasons, only the public shadow p^j has to be changed to p^{j+1}.

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This work is partialy supported by UGC grant no. F.8-9/98 (SR-I)

According to CHI-SUNG LAIH[6], in ideal situation the dynamic threshold scheme must satisfy the following characteristics:

(1) At the beginning of the time, $t_j=1$, dynamic threshold scheme, like conventional threshold scheme, provide perfect security upto the threshold value. For the master key K^j, $j=1$, converyed by the scheme, we have

Prob (K^j/given m-1 or fewer shadows and the public shadow p^j)= Prob (K^j).

(2) If the previous master keys, K^j, $j=1,\ldots,v-1$, are kept secret, the scheme also provide Shannon perfect security for the following master keys K^j, $j\geq v$. More clearly, knowing any u (even $u\geq m$) public shadows p^j, $j=1,----,u$, can not provide any information to derive any master key K^j, $j\geq v$, i.e.

Prob (any new master key $K^j, j\geq v$ /given any u public shadows $p^i, i=1,\ldots,u$) =Prob (any new $K^j, j\geq v$).

(3) Knowing any v-1 previous master keys K^j, $j=1,----,v-1$, the scheme also provides Shannon perfect security for the following new master keys $K^j, j\geq v$. That is knowing all v-1 ($\geq m$) previous master keys K^j, $j=1,----v-1$, cannot provide any information to derive new master keys K^j, $j\geq v$,i.e.

Prob (any new master key $K^j, j\geq v$/given all $K^j, j<v$ and $p^i, i=1,----,u$)=prob (any new master keys $K^j, j\geq v$).

In general, it is very difficult to design an ideal dynamic threshold scheme to satisfy the characteristics (1) to (3). Laih et al [6] defined the "relatively dynamic threshold scheme", the schemes which satisfy the two characteristics (1) and (2) above. They [6] constructed $(1, m, n, T)$ dynamic threshold scheme based on the definition of the cross- product in an n-dimensional linear space. In brief, their construction is as follows:

Assume that the total number of shadows need to be constructed be n and that m be the threshold value which works with the public shadow, p^j, to recover the single master key K^j.

SHADOW GENERATION:

For $j=1$ to T repeat steps 1-3:

Step 1 : The key generation center randomly selects $m+1$ linearly independent $(m+2)$-dimensional vectors $V_1, V_2, ---, V_m$ and V^j_{m+1}.

Step 2: The center then evaluate a new vector $U^j=(u^j_1,----u^j_{m+2})$

$=V_1 x V_2 x----x V_t x V^j_{m+1}$, and system master key K^j, at the time $t_j, 1 \le t_j \le T$ is obtained from the U^j as follows:

$$K^j = \prod_{i=2}^{m+2} abs(u_i^j)$$

The master key K^j is kept secret but the first element, $u_1^j (\neq 0)$ of the vector U^j is made public.

Step3: The n shadows Si, i=1,2----,n and the public shadow p^j are constructed by randomly selecting an $(n+1)$x$(m+1)$ matrix A^j and then executing the following operation:

$$\begin{bmatrix} S_1 \\ S_2 \\ : \\ S_n \\ p^j \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1m} & 0 \\ : & : & \ldots\ldots & : & : \\ : & : & \ldots\ldots & : & : \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nm} & 0 \\ b_1^j & b_2^j & \ldots\ldots & b_m^j & b_{m+1}^j \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ : \\ V_m \\ V_{m+1}^j \end{bmatrix}$$

The center then secretly distributes these n secret shadows S_i.

MASTER KEY RECOMPUTATION:

Knowledge of any m shadows W_i, i=1,----,m, from S_i, i=1, -----,n and p^j uniquely determines the key K^j.

First evaluate

W_1xW_2x----xW_mxp^j=$(w^j_1, w^j_2, ----, w^j_{m+1})$.

Then the master key K^j can be calculated as

$$K^j = \prod_{i=2}^{m+2} abs\left(\frac{w_i^j}{h^j}\right), \quad with \quad h = \frac{w_i^j}{u_1^j}$$

In our proposed scheme "pseudo dynamic threshold scheme" we replace public shadow p^j with a secret additional information which is supplied to KDC (key distribution center which is being discussed ahead) at the time when the

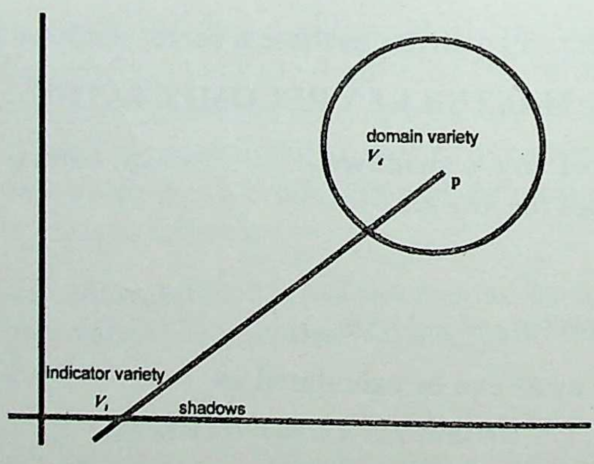
secret needs to be changed. This additional information is transferred through a secret channel or in an encrypted form. Due to this modification in the original scheme, characteristic set for the scheme has been changed.

Now there is no need to consider characteristic 2 and we shall show that our scheme satisfy characteristics 1 and 3.

2. Proposed geometrical construction of $(1,m,n,T)$ scheme: We describe the construction of $(1,m,n,T)$ dynamic threshold scheme in a simple geometrical way with the trapdoor property of discrete log. Later we shall examine whether this construction satisfies the characteristics 1 and 3 of dynamic threshold scheme.

First we define some notations taken from G.J. Simmons [9].

2-1. Geometrical notations/tools: There should be one geometric object V_i (an algebraic variety which may be a linear space of higher dimension) which can be determined by the points at general position on it and which intersects another geometric object V_d (again and algebraic variety) at a single point, say p. Though p is contained in both the objects V_i and V_d, we call V_i as indicator variety as it appears as indicating (or pointing towards) the point p in V_d, while V_d being publicly known can be thought of a domain containing all the possible values of the secret and so is called as a domain variety. We can illustrate it as:



Now it is clear from the above information that if the point of intersection p be our secret key then as soon as V_i is formed, p can be determined uniquely. Thus if we keep V_i unchanged and V_d changing, we can get different points of intersection i.e. different keys each time. This fact is the basis of our construction which is as follows:

2-2 Construction: Here we assume that all the work of computation and distribution of shadows and recomputation of the secret keys is performed in a **key distribution center (KDC)** which is not accessible to any of the participants. We also follow a hierarchy i.e. all the decisions are to be made from a higher authority (dealer) and participants are from the lower authority. KDC will be working as the mediator. We also assume that dealer and KDC have their own communication channel which may be open or secret depending upon the condition whether they are using asymmetric cryptosystem or not to pass the additional information.

We also make a little modification in our notation that instead of taking point of intersection p as secret, we take it as discrete log of the secret key i.e. if p be point of intersection then our secret point (key) is

$$y=b^p \mod q$$

where b is generally taken as the generator of the field $F(q)$ for some prime q.

Shadows Generation:

In order to construct a $(1,m,n,T)$ pseudo dynamic threshold scheme, dealer with the help of KDC generate shadows as follows:

Step 1: The dealer selects a vector space of higher dimension.

Step 2: It again select a subspace V_i of S, of dimension $(m-1)$.

Step 3: It, then, finds k lines V_d^j, $i=1,\ldots,k$, at time t_j, $1 \le t_j \le T$, of S such that the intersection of each V^j_d with V_i is a unique point p^j, $j=1$, -----k in V_i i.e. $V_i \cap V_d^j = p^j, 1 \le j \le$

Step 4: KDC stores these points and calculates secret keys as $K^j=b^{p^j} \mod. q$

Step 5: Now it selects n different points in general position, randomly and uniformly in V_i, none of which is from the already selected points p^j's and distributes one point each to the n participants.

Secret Key Recovery:

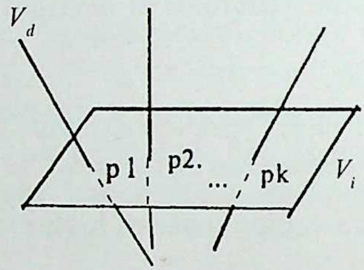
Suppose our secret be K^s, $1 \le s \le k$ at any time t_j, $1 \le t_j \le T$, then corresponding discrete log is the point p^s. The dealer puts the V_d^s in the KDC as the secret additional information. Thus whenever any m participants expose their shares to KDC, V_i can uniquely be constructed and whose intersection with V^s_d gives p^s. Now KDC calculates.

$$K^s = b^{ps} \bmod q$$

and reveals it to the participants as secret.

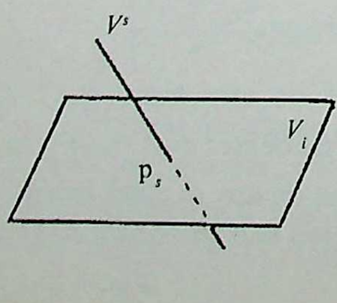
Now, if any time, dealer needs to change the secret key K^s due to some security reasons, then he will have to change only $V^s{}_d$ by some $V^j{}_d$, $j=1,----,k$, j's and thus the secret key will be changed automatically without any change in shadows.

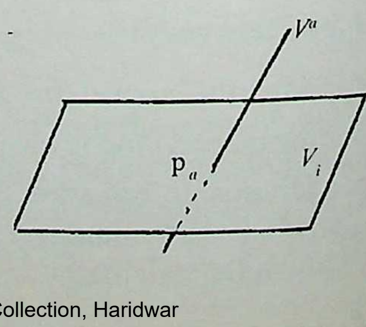
Example: Suppose we want to construct a $(1,3,n,T)$ Scheme. Now KDC will choose V_i as 2-dim object i.e. a plane and k lines $V^j{}_d$, $j=1,----,k$ each intersecting V_i at k different points p^j, $j=1,----,k$. Let the secret be K^s corresponding to point p^s, $1 \leq s \leq k$. Then higher authority will pass line V_d^s to KDC as additional information and KDC distributes n shadows to n participants as points in V_i other than p^j's. All these points are in general position i.e. no three points lie on the same common line in the plane V_i.



Now we know that a plane can uniquely be determined by any three non-collinear points on it. So whenever any three participants expose their shadows to KDC, V_i is constructed and whose intersection with $V^s{}_d$ gives p^s. After calculating $K^s = b^{p^s} \bmod q$. KDC reveals K^s to the participants.

If higher authority wants to change the secret from K^s to K^a, $1 \leq a \leq k, a, s$, then it will just pass $V^a{}_d$ to KDC and same process can be applied to get the K^a without changing any predistributed shadows.





Thus our aim is achieved in the sense that without changing shadows we can change the secret easily by changing public part $V^{\emptyset}_{d}$.

3. **Analysis and discussion:** It is quite obvious from our construction that both the characteristics 1 and 3 of relative dynamic threshold scheme are satisfied :

(1) From the above example, given any two points, every other point in V_i is equally likely to be the third point and same is the case with the point of intersection. Thus

Prob (K^i/ given any (m-1) shadows and public part)

$$=\text{Prob}\ (K^i).$$

Also if the points in V_i are selected uniformly and randomly then

$$H(p)=\log/Vi/,$$

where P denotes the shadow space and H is the entropy function.

Now we analyse the 3rd characteristic. We see that this characteristic is also fulfilled in the sense that previous secret keys are not the point of intersection themselves. And also since discrete log is a one way function it is computationally infeasible to compute point of intersection from previous $K^{i'}$ and thus V_i cannot be constructed and scheme is safe.

From information theoretical stand point we get

$$H(p^n/K^n)=H(p^n)=\log/Vi/=H(P)$$

Also $H(K^n/p^n) = 0$ and relative mutual information is

$$I(K^j;K^i,=1,\ldots.j\text{-}1) = H(K^j)\text{-}H(K^j/K^i,i\ldots.,j\text{-}1)$$

$$= H(K^j)\text{-}H(K^j)$$

$$= 0.$$

From security standpoint, it is advised to take a field as large as possible as this will make hard to guess the secret.

CONCLUSION

Thus combining the geometrical properties with one way discrete log function and with few assumptions (including alterations/modifications), we can get a dynamic threshold scheme (viz. pseudo dynamic threshold scheme) which serves in the same manner as expected from a fully dynamic threshold scheme.

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Journal of Natural & Physical Sciences, Vol. 14 (1-2) (2000) 105-110

A FIXED POINT THEOREM IN FOUR METRIC SPACES

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(Received 10-09-2000)

ABSTRACT

A fixed point theorem for maps involving four metric spaces is obtained. Special cases are discussed.

Mathematics Subject Classification : 54H25

Keywords : Fixed point, metric space.

INTRODUCTION

B. Fisher [1,2], Ng. P. Nung [3], B.D. Pant [4] and others have obtained common fixed point theorems for maps on two and three metric spaces. The purpose of this note is to obtain a related fixed point theorem for maps involving four metric spaces. Some of the known results are discussed.

MAIN RESULTS

Theorem 1. : Let (X_i, d_i), i=1,2,3,4, be complete metric spaces. If $T_1{:}X_1 \to X_2$, $T_2{:}X_2 \to X_3$, $T_3{:}X_3 \to X_4$ and $T_4{:}X_4 \to X_1$ are continuous mappings satisfying the inequalities:

$$d_1(T_4T_3T_2T_1p,\ T_4T_3T_2q) \le c \max\ \{d_1(p,\ T_4T_3T_2q),\ d_1(p,\ T_4T_3T_2T_1p),$$

$$d_2(q,\ T_1P),\ d_3(T_2q,\ T_2T_1p),\ d_4(T_3T_2q,\ T_3T_2T_1p)\};$$

$$d_2T_1(T_4T_3T_2q,\ T_1T_4T_3r) \le c \max\{d_2(q,\ T_1T_4T_3r),\ d_2(q,\ T_1T_4T_3T_2q),$$

$$d_3(r,\ T_2q),\ d_4(T_3r,\ T_3T_2q),\ d_1(T_4T_3r,\ T_4T_3T_2q)\};$$

$$d_3(T_2T_1T_4T_3r,\ T_2T_1T_4s) \le c \max\ \{d_3(r,\ T_2T_1T_4s),\ d_3(r,\ T_2T_1T_4T_3r),$$

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$$d_4(s, T_3r), d_1(T_4s, T_4T_3r), d_2(T_1T_4s, T_1T_4T_3r)\};$$

$$d_4(T_3T_2T_1T_4s, T_3T_2T_1p) \leq c \max \{d_4(s, T_3T_2T_1p), d_4(s, T_3T_2T_1T_4s),$$
$$d_1(p, T_4s), d_2(T_1P, T_1T_4s), d_3(T_2T_1p, T_2T_1T_4s)\}$$

for every $p \in X_1, q \in X_2, r \in X_3, s \in X_4$, where $0 \leq c < 1$. Then $T_4T_3T_2T_1$ has a unique fixed point $u \in X_1$, $T_1T_4T_3T_2$ has a unique fixed point $v \in X_2$, $T_2T_1T_4T_3$ has a unique fixed point $w \in X_3$ and $T_3T_2T_1T_4$ has a unique fixed point $z \in X_4$. Further $T_1u=v$, $T_2v=w$, $T_3w=z$, $T_4z=u$.

Proof. Let $x_{1,0} \in X_1$ be an arbitrary point. Define sequences $\{x_{1,n}\}$, $\{x_{2,n}\}$, $\{x_{3,n}\}$ and $\{x_{4,n}\}$ in X_1, X_2, X_3, X_4 respectively by $x_{1,n} = (T_4T_3T_2T_1)^n x_{1,0}$, $x_{2,n}=T_1 x_{1,n-1}$, $x_{3,n} = T_2x_{2,n}$ and $x_{4,n} = T_3x_{3,n}$ for $n = 1, 2, 3, \ldots$. Then

$$d_1(x_{1,n}, x_{1,n+1}) = d_1(T_4T_3T_2x_{2,n}, T_4T_3T_2T_1x_{1,n})$$
$$\leq c \max \{d_2(x_{2,n}, x_{2,n+1}), d_3(x_{3,n}, x_{3,n+1}), d_4(x_{4,n}, x_{4,n+1})\}$$

and similarly,

$$d_2(x_{2n}, x_{2,n+1}) = d_2(T_1T_4T_3x_{3,n-1}, T_1T_4T_3T_2x_{2,n})$$
$$\leq c \max \{d_3(x_{3,n-1}, x_{3,n}), d_4(x_{4,n-1}, x_{4,n}), d_1(x_{1,n-1}, x_{1,n})\},$$
$$d_3(x_{3,n}, x_{3,n+1}) = d_3(T_2T_1T_4x_{4,n-1}, T_2T_1T_4T_3x_{3,n})$$
$$\leq c \max \{d_4(x_{4,n-1}, x_{4,n})\ d_1(x_{1,n-1}, x_{1,n}), d_2(x_{2n}, x_{2,n+1})\}$$

and

$$d_4(x_{4,n}, x_{4,n+1}) = d_4(T_3T_2T_1x_{1,n-1}, T_3T_2T_1T_4x_{4,n})$$
$$\leq c \max \{d_1(x_{1,n-1}, x_{1,n}), d_2(x_{2,n}, x_{2,n+1}), d_3(x_{3,n}, x_{3,n+1})\}.$$

From these inequalities, we have

$$d_1(x_{1,n}, x_{1,n+1}) \leq c \max \{d_2(x_{2,n}, x_{2,n+1}), d_3(x_{3,n}, x_{3,n+1}),$$
$$c \max \{d_1(x_{1,n-1}, x_{1,n}), d_2(x_{2,n}, x_{2,n+1}), d_3(x_{3,n}, x_{3,n+1})\}\}$$
$$= c \max \{d_2(x_{2,n}, x_{2,n+1}), d_3(x_{3,n}, x_{3,n+1}), c\ d_1(x_{1,n-1}, x_{1,n})\}$$
$$\leq c \max \{d_2(x_{2,n}, x_{2,n+1}), c \max \{d_4(x_{4,n-1}, x_{4,n}),$$

$$d_1(x_{1,n-1}, x_{1,n}), d_2(x_{2,n}, x_{2,n+1})\}, c\, d_1(x_{1,n-1}, x_{1,n})\}$$
$$= c \max \{d_2(x_{2,n}, x_{2,n+1}), c\, d_4(x_{4,n-1}, x_{4,n}), c\, d_1(x_{1,n-1}, x_{1,n})\}$$
$$\le c \max \{c \max \{d_3(x_{3,n-1}, x_{3,n}), d_4(x_{4,n-1}, x_{4,n}), d_1(x_{1,n-1}, x_{1,n})\},$$
$$c\, d_4(x_{4,n-1}, x_{4,n}), c\, d_1(x_{1,n}, x_{1,n-1})\}$$
$$= c^2 \max \{d_1(x_{1,n-1}, x_{1,n}), d_3(x_{3,n-1}, x_{3,n}), d_4(x_{4,n-1}, x_{4,n})\}$$

It follows inductively that

$$d_1(x_{1,n}, x_{1,n+1}) \le c^{2(n-1)} \max \{d_1(x_{1,1}, x_{1,2}), d_3(x_{3,1}, x_{3,2}), d_4(x_{4,1}, x_{4,2})\}$$
$$\le c^{2(n-1)} \max \{d_2(x_{2,1}, x_{2,2}), d_3(x_{3,1}, x_{3,2}), d_4(x_{4,1}, x_{4,2})\}.$$

Similarly

$$d_2(x_{2,n}, x_{2,n+1}) \le c \max \{d_3(x_{3,n-1}, x_{3,n}), d_4(x_{4,n-1}, x_{4,n}), d_1(x_{1,n-1}, x_{1,n})\}.$$

It follows inductively that

$$d_2(x_{2,n}, x_{2,n+1}) \le c^{n-1} \max \{d_3(x_{3,1}, x_{3,2}), d_4(x_{4,1}, x_{4,2}), d_1(x_{1,1}, x_{1,2})\}.$$

Similarly

$$d_3(x_{3,n}, x_{3,n+1}) \le c^{n-1} \max \{d_4(x_{4,1}, x_{4,2}), d_1(x_{1,1}, x_{1,2}), d_2(x_{2,1}, x_{2,2})\}$$

and

$$d_4(x_{4,n}, x_{4,n+1}) \le c^{n-1} \max \{d_1(x_{1,1}, x_{1,2}), d_2(x_{2,1}, x_{2,2}), d_3(x_{3,1}, x_{3,2})\}.$$

Since $c < 1$ and the spaces are complete, $\{x_{1,n}\}$, $\{x_{2,n}\}$, $\{x_{3,n}\}$ and $\{x_{4,n}\}$ are Cauchy sequences and hence converge, say to u, v, w and z in X_1, X_2, X_3, X_4 respectively. Now by the continuity of T_1, T_2 and T_3 we have

$$\lim_{n\to\infty} x_{2,n} = \lim_{n\to\infty} T_1 x_{1,n-1} = T_1 u = v,$$

$$\lim_{n\to\infty} x_{3,n} = \lim_{n\to\infty} T_2 x_{2,n} = T_2 v = w,$$

and

$$\lim_{n\to\infty} x_{4,n} = \lim_{n\to\infty} T_3 x_{3,n} = T_3 w = z,$$

We show that u is a fixed point of $T_4T_3T_2T_1$. We have

$$d_1(T_4T_3T_2T_1u, x_{1,n}) = d_1(T_4T_3T_2T_1u, T_4T_3T_2x_{2,n})$$

$\leq c \max \{d_1(u, T_4T_3T_2x_{2,n}), d_1(u, T_4T_3T_2T_1u),$

$d_2(x_{2,n}, T_1u), d_3(T_2x_{2,n}, T_2T_1u), d_4(T_3T_2x_{2,n}, T_3T_2T_1u)\}$

$= c \max \{d_1(u, x_{1,n}), d_1(u, T_4T_3T_2T_1u), d_2(x_{2,n}, T_1u),$

$d_3(x_{3,n}, w), d_4(x_{4,n}, z)\}.$

By letting $n \to \infty$ and by the continuity of T_1, T_2 and T_3 we have

$d_1(T_4T_3T_2T_1u, u) \leq c\, d_1(u, T_4T_3T_2T_1u).$

Since $c < 1$, it follows that

$d_1(u, T_4T_3T_2T_1u) = 0$

i.e, $T_4T_3T_2T_1 u = u$.

Further

$T_1T_4T_3T_2v = T_1T_4T_3T_2T_1u = T_1u = v,$

$T_2T_1T_4T_3w = T_2T_1T_4T_3T_2v = w$

and

$T_3T_2T_1T_4z = T_3T_2T_1T_4T_3w = T_3w = z.$

Hence u is a fixed point of $T_4T_3T_2T_1$, v is a fixed point of $T_1T_4T_3T_2$, w is a fixed point of $T_2T_1T_4T_3$ and z is a fixed point of $T_3T_2T_1T_4$.

Now we prove the uniqueness. Let u and u' ($u \neq u'$) be two fixed points of $T_4T_3T_2T_1$. Then

$d_1(u, u') = d_1(T_4T_3T_2T_1u, T_4T_3T_2T_1u')$

$\leq c\max \{d_1(u, T_4T_3T_2T_1u'), d_1(u, u), d_2(T_1u', T_1u),$

$d_3(T_2T_1u', T_2T_1u), d_4(T_3T_2T_1u', T_3T_2T_1u)\}$

$= c\max \{d_2(T_1u', T_1u), d_3(T_2T_1u', T_2T_1u), d_4(T_3T_2T_1u', T_3T_2T_1u)\}.$

Now

$d_2(T_1u, T_1u') = d_2(T_1T_4T_3T_2T_1u, T_1T_4T_3T_2T_1u')$

$\leq c\max \{d_3(T_2T_1u', T_2T_1u), d_4(T_3T_2T_1u', T_3T_2T_1u),$

$d_1(u', u)\}$

$\leq c \max \{d_3(T_2T_1u', T_2T_1u), d_4(T_3T_2T_1u', T_3T_2T_1u)\}$

and

$d_3(T_2T_1u', T_2T_1u) = d_3(T_2T_1T_4T_3T_2T_1u', T_2T_1T_4T_3T_2T_1u)$

$\leq c\max \{d_3(T_2T_1u, T_2T_1u'), d_4(T_3T_2T_1u', T_3T_2T_1u),$

$d_1(u', u), d_2(T_1u', T_1u)\}$

$= cd_4(T_3T_2T_1u', T_3T_2T_1u).$

Also

$d_4(T_3T_2T_1u', T_3T_2T_1u) \leq c\max \{d_4(T_3T_2T_1u', T_3T_2T_1u), d_1(u, u'),$

$d_2(T_1u, T_1u'), d_3(T_2T_1u, T_2T_1u')\}$

$= cd_1(u, u').$

Hence

$d_1(u, u') \leq c \max\{c \max\{d_3(T_2T_1u', T_2T_1u), d_4(T_3T_2T_1u', T_3T_2T_1u)\},$

$d_3(T_2T_1u', T_2T_1u), d_4(T_3T_2T_1u', T_3T_2T_1u)\}$

$\leq cd_4(T_3T_2T_1u', T_3T_2T_1u) \leq c^2d_1(u,u').$

This implies $u=u'$, u is the unique fixed point of $T_4T_3T_2T_1$.

Similarly, v is the unique fixed point of $T_1T_4T_3T_2$, w is the unique fixed point of $T_2T_1T_4T_3$ and z is the unique fixed point of $T_3T_2T_1T_4$. This completes the proof.

REMARKS

1. If we consider $X_1=X_4=X$, $X_2=Y$, $X_3=Z$ and $d_1=d_4=d$, $d_2=\rho$, $d_3=\sigma$, $T_1=T$, $T_2=S$, $T_3= R$ and $T_4=I$ is the identity mapping from X into itself, then we obtain the main result of Ng Peng Nung [3].

2. If we consider $X_1=X_3=X_4=X$, $X_2=Y$, $d_1=d_3=d$, $d_2=\rho$ and T_3 and T_4 are the identity mappings from X into itself, then we obtain a fixed point theorem of Brian Fisher [1].

Theorem 2. Let (X_i, d_i), $i= 1, \ldots, k$, be complete metric spaces. If

$T_i: X_i \to X_{i+1}$, $i = 1, \ldots (k-1)$ and $T_k: X_k \to X_1$ are continuous mappings satisfying the inequalities:

$$d_1(T_kT_{k-1}T_{k-2}\ldots T_2T_1x,\ T_kT_{k-1}T_{k-2},\ \ldots,\ T_2y)$$
$$\leq c\max\{d_1(x,\ T_kT_{k-1}T_{k-2}\ldots T_2y),\ d_1(x,T_kT_{k-1}T_{k-2}\ldots\ T_2T_1x),$$
$$d_2(y,T_1x),\ldots,\ d_k(T_{k-1}T_{k-2}\ldots T_2y,T_{k-1}T_{k-2}\ldots T_2T_1x)\}$$

for all $x \in X_1$, $y \in X_2$,

$$d_i(T_{i-1}T_{i-2}\ldots T_1T_kT_{k-1}\ldots T_ip,\ T_{i-1}\ T_{i-2}\ldots T_1T_kT_{k-1}\ldots T_{i+1}q)$$
$$\leq c\max\{d_i(p,T_{i-1}T_{i-2}\ldots T_1T_kT_{k-1}\ldots T_{i+1}q),\ d_i(p,T_{i-1}T_{i-2}\ldots T_1T_k\ldots T_ip),$$
$$d_{i+1}(q,T_ip),\ d_{i+2}(T_{i+1}q,T_{i+1}T_ip),\ldots,$$
$$d_k(T_kT_{k-1}\ldots T_{i+1}q,\ T_kT_{k-1}\ldots T_{i+1}T_ip),$$
$$d_1(T_1T_kT_{k-1}\ldots\ldots T_{i+1}q,\ T_1T_kT_{k-1}\ldots\ldots T_{i+1}T_ip),\ldots\ldots,$$
$$d_{i-1}(T_{i-2}T_{i-3}\ldots\ldots T_{i+1}q,\ T_{i-2}T_{i-3}\ldots\ldots T_{i+1}T_ip)\}$$ for all $p \in X_i$, $q \in X_{i+1}$,

$i=2,3,4,\ldots\ldots,(k-1)$ and $0 \leq c < 1$.

Then $T_kT_{k-1}\ldots\ldots T_2T_1$ has a unique fixed point $u_l \in X_l$,

$T_{i-1}\ T_{i-2}\ldots\ldots T_1T_kT_{k-1}\ldots\ldots T_i$ has a unique fixed point $u_i \in X_i$, $i = 2,3,\ldots\ldots\ldots,(k-1)$. Further $T_iu_i = u_{i+1}$, $i = 1,2,\ldots\ldots,(k-1)$ and $T_ku_k = u_l$.

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Journal of Natural & Physical Science, Vol. 14 (1-2) (2000) 111-116

ON CERTAIN TRANSFORMATIONS OF BASIC HYPERGEOMETRIC FUNCTIONS

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(Received 03.05.1999 and on revision 25.09.2000)

ABSTRACT

We make use of summation formulae due to Srivastava [3] to establish certain expansions of basic hypergeometric series. Some particular cases are also discussed.

Keywords and phrases : Hyperegometric series, Basic hypergeometric series, Summation formulae etc.

Mathematics subject classificatin (AMS) : 33A30.

INTRODUCTION

For real or complex q, $|q| < 1$ let

$$(a,q)\mu = \prod_{j=0}^{\infty}\left[\frac{1-aq^{j}}{1-aq^{\mu+j}}\right] \tag{1.1}$$

for arbitrary a and μ so that

$$(a,q)n = \begin{cases} 1 \quad if \quad n=0 \\ (1-a)(1-aq).......(1-aq^{n-1}) \\ \quad where \quad n=1,2,3........ \end{cases} \tag{1.2}$$

and

$$(a;q)_{\infty} = \prod_{j=0}^{\infty}\left(1-aq^{j}\right) \tag{1.3}$$

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For convenience, we shall write $[a]_n$ to mean $(a\,;q)_n$. If the base is not q but say p then we shall mention it explicitly as $(a\,;p)_n$.

Recently Denis [1] obtained certain expansions of basic double hypergeometric series defined as

$$\phi^{p:h:u}_{l:m:n}\begin{bmatrix}(\alpha_p):(\beta_h)\,;(\delta_u)\,; & \\ & q\,;\,x,y\\ (\lambda_l):(\mu_m)\,;(\nu_n)\,; & \end{bmatrix}$$

$$=\sum_{r,s=0}^{\infty}\frac{[(\alpha_p)]_{r+s}\;[(\beta_n)]_r\;[(\delta_u)]_s\,x^r y^s}{[(\lambda_t)]_{r+s}\;[(\mu_m)]_r\;[(\nu_n)]_s[q]_r[q]_s} \tag{1.4}$$

where $|x|<1$, $|y|<1$. The parameters (α_p) denote the sequence of parameters $\alpha_l, \alpha_2, \ldots\ldots \alpha_p$ and $p=0$ shall indicate the absence of the parameters (α_p).

In the course of our analysis we require the following known results [2,iv, 11, p 2 48] :

$$\sum_{n=0}^{\infty}\frac{[a]_n}{[q]_n}x^n=\prod\begin{bmatrix}ax;\\ x\end{bmatrix} \tag{1.5}$$

$$\sum_{r,s=0}^{\infty}\frac{[\mu]_{r+s}\;[\alpha]_r\;[q^{-m}]_r\;[\beta]_s\;[q^{-n}]_s\,q^{r+s}}{[\alpha/\beta q^{1-m}]_r\;[\mu]_r\;[\beta/\alpha\; q^{1-n}]_s\;[\mu]_s\;[q]_r\;[q]_s} \tag{1.6}$$

$$=\frac{[\mu]_{m+n}[\beta]_m[\alpha]_m}{[\beta/\alpha]_m[\mu]_m[\alpha/\beta]_n[\mu]_n} \tag{1.6}$$

$m,n, = 0,1,2,\ldots\ldots$

$$\sum_{r,s=0}^{\infty}\frac{[\mu/\alpha]_r[\mu/\beta]_r[q^{-m}]_r[\alpha]_s[\beta]_s[q^{-n}]_s\,q^{r+s}}{[\mu]_{r+s}[\mu q^{1-m}/\alpha\beta]_r[\alpha\beta/\mu\; q^{1-n}]_s[q]_r[q]_s}$$

$$= \frac{[\alpha]_m [\mu/\alpha]_m [\beta]_m [\mu/\beta]_n}{[\mu]_{m+n} [\alpha\beta/\mu]_m [\mu/\alpha\beta]_n} \tag{1.7}$$

where $m,n, = 0,1,2,....$ The summation formulae (1.6) and (1.7) are obtained by Srivastava [3,P.7,eq. (3.5) and (3.8)]

Here we make use of (1.6) and (1.7) to establish certain expansions of basic hypergeometric series defined by (1.4). Some particular cases are also discussed.

MAIN RESULTS

In this section, we establish following two main results and discuss their special cases

$$\phi^{P+1:M+1;M'+1}_{t:N+2:N'+2}\left[\begin{matrix}(a_p),\mu:(b_m),\beta;(b'_{m'})\alpha; \\ (c_t):\beta/\alpha,\mu,(d_N);\alpha/\beta,\mu,(d'_{N'})\end{matrix}\; q\;;\, x,y\right]$$

$$= \sum_{r,s=0}^{\infty} \frac{[\mu]_{r+s}[(a_p)]_{r+s}[(b_M)]_r[(b'_{M'})]_s}{[(c_t)]_{r+s}[(d_N)]_r[(d'_{N'})]_s[\mu]_r[\mu]_s} \frac{[\alpha]_r[\beta]_s}{[\beta/\alpha]_r[\alpha/\beta]_s[q]_r[q]_s}(\beta x/\alpha)^r\ (\alpha y/\beta)^s$$

$$\phi^{P:M+1;M'+1}_{t:N+1;N'+1}\left[\begin{matrix}(a_p)q^{r+s}:\beta/\alpha,(b_M)q^r;\alpha/\beta,(b'_{M'})\ q^s \\ (c_t)q^{r+s}:\beta/\alpha\ q^r,(d_N)q^r;\alpha/\beta\ q^s,(d'_{N'})\ q^s;\end{matrix}\; x,y\right] \tag{2.1}$$

and

$$\phi^{P:M+2,M'+2}_{t+1:N+1,N'+1}\left[\begin{matrix}(a_p):\alpha,\beta,(b_M);\mu/\alpha,\mu/\beta,(b'_{M'}); \\ \mu,(c_t):\alpha\beta/\mu,(d_N);\mu/\alpha\beta,(d'_{N'});\end{matrix}\; x,y\right]$$

$$= \sum_{r,s=0}^{\infty} \frac{[(a_p)]_{r+s}[(b_M)]_r[(b'_{M'})]_s[\mu/\alpha]_r}{[\mu]_{r+s}[(c_t)]_{r+s}[(d_N)]_r[(d'_{N'})]_s} \frac{[\mu/\beta]_r[\alpha]_s[\beta]_s}{[\alpha\beta/\mu]_r[\mu/\alpha\beta]_s[q]_r[q]_s}(a\beta x/\mu)^r(\mu y/\alpha\beta)^s$$

$$\times\ \phi^{P:M+1;M'+1}_{t:N+1;N'+1}\left[\begin{matrix}(a_p)q^{r+s}:\alpha\beta/\mu,(b_M)q^r;\mu/\alpha\beta,(b_{M'})q^s; \\ (c_t)q^{r+s}:(\alpha\beta/\mu)q^r,(d_N)q^r,\mu/\alpha\beta\ \ q^s,(d'_{N'})q^s\end{matrix}\ x,y\right] \quad (2.2)$$

Proof of (2.1) : Making an appeal to (1.4) and (1.6), we derive

$$\phi^{P+1:M+1;M'+1}_{t:N+2;N'+2}\left[\begin{matrix}(a_p),\mu:(b_M),\beta;(b'_{M'}),\alpha; \\ (c_t):\beta/\alpha,\ \mu,(d_N);\alpha/\beta,\ \mu,(d'_{N'});q,\end{matrix}\quad x,y\right]$$

$$=\sum_{m,n=0}^{\infty}\frac{[(a_p)]_{m+n}[(b_M)]_m[(b'_{M'})]_n\,x^m y^n}{[(c_t)]_{m+n}[(d_n)]_m[(d'_{N'})]_n[q]_m[q]_n}$$

$$\times\sum_{r,s\geq 0}\ \frac{[\mu]_{r+s}[\alpha]_r[\beta]_s[q^{-m}]_r[q^{-n}]_s q^{r+s}}{[(\alpha/\beta)q^{1-m}][\mu]_r[(\beta/\alpha)q^{1-n}]_s[\mu]_s[q]_r[q]_s} \quad (2.3)$$

which on replacing m by $m+r$ and n by $n+s$ in the above relation and interchanging the order of double summations, gives (2.1) after some simplification.

Proof of (2.2) : It follows similarly with the help of (1.7) in place of (1.6).

Particular Cases :

(I) For $p=N=N'=0,\ t=1,\ c_1=\mu\ \ M=M'=2,$

$b_1=\beta/\alpha,\ b'_2=\alpha/\beta,\ b_2=b'_2=\mu$ and using (1.5) the result (2.1) gets reduced to

$$\sum_{r,s=0}^{\infty}\ \frac{[\alpha]_r[\beta]_s(\beta x/\alpha)^r(\alpha y/\beta)^s}{[q]_r\ [q]_s}\ \phi^{(3)}[\beta/\alpha,\alpha/\beta;\mu q^r,uq^s,\mu q^{r+s},x,y] \quad (2.4)$$

$$=\prod\begin{bmatrix}\beta x, & \alpha y \\ x, & y\end{bmatrix}$$

where $\phi^{(3)}$ is basic Appell function.

(II) Putting $P = 0, t = N = N' = 1.d_1, = \beta, d_1' = \alpha, c_1 = \mu$,

$M = M' = 2, b_1 = \beta/\alpha\, b_1' = \alpha/\beta$ and $b_2 = b_2' = \mu$ in (2.1) then in view of (1.4), we have

$$\sum_{r,s=0}^{\infty} \frac{[\alpha]_r [\beta]_s (\beta x/\alpha)^r (\alpha y/\beta)^s}{[\beta]_r [\alpha]_s [q]_r [q]_s} \phi^{0:2;2}_{1:1;1}\left[\begin{matrix} ---:\beta/\alpha, \mu q^r; \alpha/\beta, \mu q^s; \\ \mu q^{r+s} : \beta q^r, \alpha q^s; \end{matrix} q, x, y\right]$$

$$= \frac{1}{[x]_\infty [y]_\infty} \tag{2.5}$$

(III) For $P = t = N = N' = 0, M = M' = 1, b_1 = \alpha\beta/\mu, b_1' = \mu/\alpha\beta$ and on account of (1.5) the result (2.2) gets reduced to

$$\phi^{(3)}\left[\alpha, \mu/\alpha ; \beta, \mu/\beta; \mu; x, y\right]$$

$$= \prod\begin{bmatrix} \alpha\beta x/\mu & , & \mu y/\alpha\beta; \\ & & \\ x & , & y \end{bmatrix} \phi^{(3)}\left[\mu/\alpha, \alpha; \mu/\beta, \beta; \mu ; \alpha\beta x/\mu, \mu y/\alpha\beta\right] \tag{2.6}$$

As $q \to 1$, (2.6) gives the transformation for Appell function F_3 in the form

$$F_3[\alpha, \mu-\alpha ; \beta , \mu-\beta ; \mu ; x , y] = (1-x)^{\mu-\alpha-\beta} (1-y)^{\alpha+\beta-\mu}$$

$$\times F_3[\mu-\alpha, \alpha; \mu-\beta, \beta; \mu ; x, y]$$

ACKNOWLEDGEMENT

The author is greatful to the referee for fruitful suggestions.

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Journal of Natural & Physical Sciences, Vol. 14 (1-2) (2000) 117-124

GOMPERTZ EQUATION : THERMODYNAMIC MODELLING OF BIOLOGICAL GROWTH

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(Received 24.04.2000 and after revision 07.10.2000)

ABSTRACT

The object of this paper is to study the importance of Gompertz equation in the nonequilibrium thermodynamic modelling of biological growth.

Mathematics Subject Classification (2000) : 92A17

Key Words : Gompertz equation, Thermodynamic flux and force, Stationary and Quasi-stationary states, Entropy- Production.

INTRODUCTION

Growth is a fundamental process of living organism. The growth, which is a consequence of the biochemical process of metabolism is associated with a number of chemical reactions. The first attempt to the physico- chemical theory of growth based on thermodynamic processes is due to Priogogine and Weame [10]. This theory was later modified and developed by Zotin and Zotina [1,2] Lurie and Wagensberg [8,9]. There are some differences of opinions on the characteristics of growth. Some are of the opinions that biological growth, particularly embryonic growth is a linear procdess [1,8,9]. Some others hold the view that the growth is a nonlinear process [7].

There are many model equations describing growth process. A basic model equation describing a wide range of growth such as embryonic growth, tumour growth, growth of animal, men and plants etc. is the Gompertz equation.

In deriving Gompertz-equation, it is assumed that the substrate is nonlimiting so that the growth machinery is always saturated with substrate; the quantity of growth machinery is proportional to the mass (or weight) $m\ (t)>0$ with a constant of proportionality $\mu(t)>0$; the effectiveness $\mu(t)$ the growth machinery decays with time t according to the first order kinetics giving exponential decay

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with constant $\alpha>0$ and $a = \mu(o)$; this decay is due to development of mass $m(t)$ with development parameter α. Now we can formalize above concepts as

$$\frac{1}{m(t)}\frac{dm(t)}{dt} = \mu(t) \tag{1.1}$$

where the specific growth parameter $\mu(t)$ is governed by

$$\frac{1}{\mu(t)}\frac{d\mu(t)}{dt} = \alpha \tag{1.2}$$

Integrating equation (1.2) w.r. to time, we get

$$\mu(t) = a\ e^{-\alpha t},\quad a = \mu(0) \tag{1.3}$$

Putting equation (1.3) in equation (1.1) we get Gompertz equation [12]

$$\frac{dm}{dt} = a\ e^{-\alpha t} m \tag{1.4}$$

whose solution for time from 0 to t is of the form

$$m = m_0\, e^{\frac{a}{\alpha}\left(1-e^{-\alpha t}\right)},\quad m(0) = m_0 > 0 \tag{1.5}$$

where m is the mass (or weight) of an organism at any time t, a is the growth rate at time t=0 and α is the development parameter.

The validity of equation (1.4), which was established by Gompertz [13] for the description of mortality curves, was shown for growth of Cows [14] and for molluscs [15], but also for the growth of tumour [17] or embryo [7]. Although the equation (1.4) is of wide validity, there is a difficulty in taking it as the model equation in developing non-equilibrium thermodynamic theory of growth. The object of the paper is to remove this drawback and to suggest two linear thermodynamic theories of growth on the basis of the model equation (1.4). The first is based on the linearized form of the model equation (1.1) and the second is based on a slight modification of the model equation (1.1) and proceeding in analogy with a nonliner chemical reaction.

BIOLOGICAL GROWTH :A LINEARISED THERMODYNAMIC MODEL

A fundamental problem connected with the nonequilibrium thermodynamic model of biological growth is the proper choice of thermodynamic fluxes and forces. For simplicity we take a single thermodynamic flux J and a thermodynamic force X to represent the growth. We take the flux J as the specific rate of change of mass (weight).

$$J = \frac{I}{M}\frac{dm}{dt} \tag{2.1}$$

So according to the equation (1.1) we have ,

$$J = \frac{I}{M}\frac{dm}{dt} = ae^{-\alpha t} \tag{2.2}$$

Before we specify the corresponding thermodynamic force X, let us specify a difficulty encountered with the Gompertz equation (1.4). The equation (1.4) shows that the stationary or steady state of growth of a living system is reached after an infinitely large time. This is practically impossible for all living organisms, having limited growth and finite life span. It shows that for a realistic picture of growth and development there must be some restriction on the order of the exponential αt [5]. To resolve this difficulty faced with the Gompertz equation we assume that

$$|\alpha t| << 1 \tag{2.3}$$

and linearize the r.h.s. of (1.4) or (2.1). The condition (2.3) requires that either the time t must be very small implying a system not far from the embryonic or germinating state or very small value of the development parameter α. Under this restriction the equation (2.2) approximates to :

$$J = \frac{1}{m}\frac{dm}{dt} = a\alpha\left(\frac{1}{\alpha} - t\right) = a\alpha\left(t_m - t\right) \tag{2.4}$$

Where t_m=(1/ α), the time of stationary state. If we take the thermodynamic force X proportionhal to (t_m-t) : X=k(t_m-t) where t is assumed to be small and k is a constant to adjust the dimension of the force X, then the relation (2.4) reduces to the linear phenomenological relation of non - equilibrium thermodynamics.

$$J = \frac{a\alpha}{k} X = LX \tag{2.5}$$

where $J = \dfrac{a\alpha}{k}$ is the phenomenological coefficient. The linear approximation (2.4), is justified by two facts. The first is that the linear approximation makes the equation amenable to the theory of linear thermodynamics of irreversible processes, and secondly, it removes the infinitely large time to reach the stationery state. Above all, the linear model (2.4) is able in explaining a wide variety of biological phenomena related to growth and development of living organisms [1]. It is in fact , the starting or basic equation obtained from empirical consideration in an early development of nonequilibrium thermodynamics of biological growth by Zotin and Zotina [1]. In the present paper our starting equation is Gompertz equation and its linear approximation. from equation (2.3) we ave

$$\frac{dm}{dt} = a\alpha(t_m - t)m \tag{2.6}$$

Integrating (2.7) and using the condition $t=t_m, m=m_0$ we have

$$\ell n \frac{m}{m_0} - = \frac{-a\alpha}{2}(t_m - t)^2$$

$$\text{or,} \quad m = m_0 e^{-(a\alpha/2)}(t_m - t)^2 \tag{2.7}$$

which follows a Gaussian or normal distribution symmetrical about the stationary age t_m The mass (or weight) of the organism increases upto time t_m and reaches its maximum value $m = m_0$and after that then weight begins to decay.

The gaussian distribution of biomass (or weight) resulting from the linear equation in Zotin and Zotina (2.6) is satisfactorily correlated with experimental data [16]. Besides, Zotin and Zotina [1] in their paper had cited a number of experimental results of growths which could will be modelled by the growth equation (2.6)

A MODIFIED THERMODYNAMIC MODEL

In this section we shall modify the Gompertz equation (1.4) and justify the linear growth model from the consideration of chemical kinetics. In equation (2.1) we have defined thermodynamic flux J by specific growth rate

$$J = \frac{1}{m}\frac{dm}{dt} \tag{3.1}$$

Here we shall adopt a different process to define the thermodynamic flux J and force X in analogy with chemical kinetics . This type of analogy biological growth with chemical reaction was made by Lurie and Wagensberg [8]. For this we modify the equation (2.1) by substraction of a constant term of the form :

$$J' = \frac{1}{m}\frac{dm}{dt} = ae^{-\alpha t} - ae^{-\alpha t_0} \tag{3.2}$$

where t_0 is the fixed time point. This is motivated by the definition of thermodynamic force which should be a difference (or gradient) function. We can reduce (3.2) to the form :

$$J' = -ae^{-\alpha t_0}\left(1 - e^{-\alpha(t-t)}\right) = J_0\left(1 - e^{-\alpha(t-t)}\right) \tag{3.3}$$

where $J_0 = -ae^{-\alpha t_0}$ is a constant. The equation (3.3) is similar to an important expression of the chemical rate [6]

$$V = V_0\left(1 - e^{-(A/RT)}\right) \tag{3.4}$$

which is valid for nonequilibrium state far from equilibrium [6]. Here V is the rate (thermodynamic flux) and A is the affinity (thermodynamic force) of chemical reaction. If the reacting system is close to equilibrium and chemical affinity is so small that

$$\left|\frac{A}{RT} << 1\right| \tag{3.5}$$

Then we get the linear relation

$$V = \frac{V_0 A}{RT} = L_0 A \tag{3.6}$$

In analogy with chemical reaction and assuming that the macroscopic level of the description of biological growth and development is the result of numerous chemical reactions associated with metabolism of a living system, we may write

$$J' = J_0\left(1 - e^{-\alpha' X'}\right) \tag{3.7}$$

comparison (3.3) and (3.7) sueggests to take.

$$\alpha' X' = \alpha(t - t_0) \quad \text{or} \quad X' = \frac{\alpha}{\alpha'}(t - t_0) = k(t - t_0) \tag{3.8}$$

as the thermodynamic force . The relation (3.7) is nonlinear between the flux J' and force X' It reduces to the linear phenomenological relation of irreversible thermodynamics

$$J' = aJ_0X' = L'X' \tag{3.9}$$

provided

$$|\alpha' X'| << 1 \tag{3.10}$$

STATIONARY OR QUASI-STATIONARY STATES

If the termodynamic force X' is so large that the condition (3.10) is not satisfied, then the validity of linear irreversible thermodynamics for biological growth is questionable. However, the problem can be solved in other ways. We assume that the total growth of the living organism during the time interval (o,t) can be splitted into a certain number of elementary growths, having each affinity or thermodynamic force X_i' sufficiently small that the linear penomenological laws hold, that is

$$|\alpha' X_i'| << 1 \tag{4.1}$$

$$\text{and} \quad J_i' = L_{ii}' X_i' \tag{4.2}$$

Let us represent te initial growth at t=0 by I and final growth at t by F. Then the whole growth process during the time interval (o,t) can be represented in the form (in analogy with cemical equation).

$$I \rightarrow F \tag{4.3}$$

We assume that the whole growth process (4.3) proceeds in a number if successive steps :

$$\begin{array}{ll} I \rightarrow N & \text{Step (1)} \\ N \rightarrow O & \text{Step (2)} \\ \dots\dots\dots & \\ P \rightarrow F & \text{Step (r)} \end{array} \tag{4.4}$$

For the total growth process $P \rightarrow F$, we can introduce total affinity (or thermodynamic force) X' and thermodynamic flux J' from the definition (3.8) it follows that

$$\alpha' X' = k\left[(t_1 - 0) + (t_2 - t_1) + \dots\dots(t_r - t_{r-1})\right] ; (t_r = t)$$

$$\text{or,}\quad \alpha' X' = X_1' + X_2' + \ldots\ldots X_r' \tag{4.5}$$

If the intermediate states,N,O,....., P are unstable, then according go the well known kinetic method of quasi-stationary states

$$J_1' = J_2' = \ldots = J_r' = J' \tag{4.6}$$

is established after a short lapse of time [11]. Then from (4.2) and (4.6) we have

$$J' = L_{11}' X_1' = L_{22}' X_2' = \ldots\ldots = L_{rr}' X_r' \tag{4.7}$$

and from (4.5) we have then

$$X' = \sum_i X_i' = J'\left(\sum_i \frac{1}{L_{ii}}\right) = \frac{J'}{L'} \tag{4.8}$$

where

$$\frac{1}{L'} = \sum_i \frac{1}{L_{ii}} \tag{4.9}$$

hence we have the linear phenomenological relation

$$J' = L'X' \tag{4.10}$$

Thus the overall thermodymanic flux and force satisfy the linear phenomenological relation (4.10)provided the condition (4.1) that is,

$$|\alpha' X_i'| << 1 \tag{4.11}$$

is satisfied although the relation

$$|\alpha' X'| << 1 \tag{4.12}$$

may not hold for the total (or overall) thermodynamic force. The above analysis has led to two important conclusions. Firstly, it justifies the taking of the thermodynamic force proportional to the time difference discussed in section (2). Secondly, it justifies the validity if linear thermodynamics for stationary and quasi-stationary states of the growth process inspite of the nonlinear relation between the total (or overall) thermodynamic flux and force.

CONCLUSION

The paper is a modification and correction of some eariler works [5] on the study of importance of Gompertz equation in the thermodynamic modelling of biological growth. The modelling on the basis of Gompertz equation involves a difficulty with respect to the stationary state of the system. This has been resolved by linearising the Gompertz equation and

developing a linear irreversible thermodynamics of growth in analogy with chemical kinetics. The theory so developed, thus justifies the linear growth model equation of Zotin and Zotina [1] which they assumed empirically to develop the thermodynamics of biological growth.

ACKNOWLEDGEMENT

We wish to thank Prof. C.G. Chakrabarti, S.N. Bose Professor of Applied Mathematics,Calcutta university for the help in the preparation of this paper. We also wish to thank the learned referee for the comments and valuable suggestions which greatly helped the exposition of the paper.

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Journal of Natural & Physical Science, Vol. 14 (1-2) (2000) 125-128

ON A CLASS OF BILATERAL GENERATING FUNCTION FOR THE GENERALISED HERMITE POLYNOMIALS FROM A VIEW POINT OF LIE-GROUP.

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(Received 22.03.1999 and after revision 28.11.2000)

ABSTRACT

A new class of bilateral generating functions for the generalised Hermite polynomials $H_n^{\mu}(x)$, when n is even, are obtained from a view point of Lie-group. Known bilateral generating functions are obtained as a particular case.

AMS subject classification : (1991) 33 C 45
Key word and phrases : Bilateral generating function : special function.

INTRODUCTION

The generalised Hermite polynomials $H_n^{\mu}(x)$ for n even, are defined by the differential equation [2].

$$xD^2 H_n^{\mu}(x) - (2x^2 - \mu)DH_n^{\mu}(x) + 2nxH_n^{\mu}(x) = 0 \text{ where } D = \frac{d}{dx}. \tag{1.1}$$

Applying Weisner's Lie-group theoretic method [3], we obtained the following operator for $H_n^{\mu}(x)$ when n is even, by giving a suitable interpretation to n :

$$R = y\frac{\partial}{\partial x} - \frac{(2x^2 - \mu)y}{x} \tag{1.2}$$

such that

$$R\left[H_n^{\mu}(x)y^n\right] = -2H_{n+1}^{\mu}(x)y^{n+1} \tag{1.3}$$

The extended form of the transformation group is

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$$\cdot \exp wR\left[f(x,y)\right] = \left(1+\frac{wy}{x}\right)^{\mu} e^{-wy(wy+2x)} f(x+wy, y) \tag{1.4}$$

where $f(x,y)$ is an arbitrary differentiable function,

The object of the present paper is to derive a general class of bilateral generating functions by employing Lie-group theoretic method. Actually our result can be put in the form of a theorem as follows :

Theorem : If there exists a unilateral generating function of the form

$$F(x,t) = \sum_{n=0}^{\infty} a_n H_n^{\mu}(x) t^n \tag{1.5}$$

where a_n is a suitable arbitrary constant, then the following class of bilateral generating functions will hold :

$$\left(1+\frac{y}{x}\right)^{\mu} e^{2xy-y^2} F(x-y,\ ty) = \sum_{p=0}^{\infty}\sum_{n=0}^{p} \frac{a_n}{(p-n)!} (2)^{p-n} H_p^{\mu}(x) y^p t^n \tag{1.6}$$

The importance of the result (1.6) is that whenever one knows a generating function of the type (1.5) for a particular value of a_n, then the corresponding bilateral generating functions can at once be written down from (1.6). Thus, one can derive a large number of bilateral generating functions by assigning different values to a_n.

PROOF OF THE THEOREM

Let

$$F(x,t) = \sum_{n=0}^{\infty} a_n H_n^{\mu}(x) t^n \tag{2.1}$$

Replacing t by ty, we have

$$F(x,ty) = \sum_{n=0}^{\infty} a_n H_n^{\mu}(x) t^n y^n . \tag{2.2}$$

Now, let exp (wR) act on both sides of (2.2)

$$\exp(wR)F(x,ty) = \exp(wR)\sum_{n=0}^{\infty} a_n H_n^{\mu}(x)y^n t^n. \tag{2.3}$$

The left hand side of (2.3) becomes

$$\left(1+\frac{wy}{x}\right)^{\mu} e^{-wy(wy+2x)} F(x+wy,\ ty)$$

on the other hand, right hand side of (2.3) reduces to

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{a_n w^p (-2)^p}{p!} H_{n+p}^{\mu}(x) y^{n+p} t^n$$

Equating, we obtain

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{a_n w^p (-2)^p}{p!} H_{n+p}^{\mu}(x) y^{n+p} t^n = \left(1+\frac{wy}{x}\right)^{\mu} e^{-wy(wy+2x)} F(x+wy,\ ty)$$

or,
$$\sum_{p=0}^{\infty}\sum_{n=0}^{p} \frac{a_n w^{p-n} (-2)^{p-n}}{(p-n)!} H_p(x) y^p t^n = \left(1+\frac{wy}{x}\right)^{\mu} e^{-wy(wy+2x)} F(x+wy,\ ty)$$

Now, by choosing $w = -1$, we arrive at (1.6).

APPLICATION

If we put $\mu = 0$, we obtain

$$\sum_{p=0}^{\infty}\sum_{n=0}^{p} \frac{a_n (2)^{p-n}}{(p-n)!} H_p(x) y^p t^n = e^{2xy-y^2 F(x-y,\ ty)}$$

which can be well compared to a result of Chatterjea [1] derived by a different operational method.

ACKNOWLEDGEMENT

We are indebted to Dr. D.K. Basu, Reader, Department of Mathematics, Bangabasi Evening college, Calcutta for his guidance in the preparation of this paper.

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Journal of Natural & Physical Science, Vol. 14 (1-2) (2000) 129-134

KINETICS AND MECHANISM OF PERIODATE OXIDATION OF 2,4 DIMETHYLANILINE IN ACETONE- WATER MEDIUM

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(Received : 25.09.2000 and after revision : 18.12.2000)

ABSTRACT

Results of kinetic studies of the sodium metaperiodate oxidation of 2,4 dimethylaniline in acetone- water medium are discussed. A mechanism is proposed for the formation of 2,4 dimethyl-1, 2-benzoquinone, which has been isolated and characterized by us as the major product.

Key word : 2,4 dimethylaniline, Oxidation by periodate ion.

INTRODUCTION

The kinetic studies made on the non-malapradian oxidation of aromatic amines by periodate ion are rather few [1-5]. In continuation to our earliar studies [6-7], the results of the periodate oxidation of 2,4 dimethylaniline (DMA) in acetone-water medium are being reported in the present paper.

EXPERIMENTAL

DMA and sodium meta periodate of E.Merck A. R. grade were used after Zn-dust distillation/recrystallization. Doubly distilled water and other chemicals of A.R. grade were used. Thiel, Schultz and Koch buffer [8] was used for maintaining the pH of reaction mixtures.

The progress of reaction was followed by recording absorbances of light organge colored reaction mixture on Shimadzu double beam spectrophotometer, UV-150-02 at the λ max 475 nm keeping the pH at 5.0 during the period in which the λ max did not change. Initial rates in terms of $(dA/dt)_i$ were evaluated by plane mirror method while the pseudo first order rate constant k_1, was calculated by using Guggenheim method. The second order rate constants k_2 were obtained by dividing the k_1 by [S] where S is the reactant taken in excess. The stoichiometry of the reaction was determined iodimetrically.

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RESULTS AND DISCUSSION

Reaction mixture (DMA : $NaIO_4$ = 1:10) was filtered after 30 hours, filtrate extracted with petroleum ether and a major red colored component was separated from this extract by preparative T.L.C. Its melting point was found to be 69°C (lit. [9] 68.0°C). This compound responded positively for a quinone [10]. λ max for this compund in C_2H_5OH were found to be 250 nm and 410 nm which are in good agreement for the values reported in literature [11] for 2,4-dimethyl 1,2-benzoquinone.

I.R. spectrum (in KBr) showed bands at 3054 cm^{-1} (S), 2925cm^{-1} (S), 2854cm^{-1} (S), 1660cm^{-1} (S), 1514cm^{-1} (S), 1455cm^{-1} (S), 750cm^{-1} (S) and 810cm^{-1} (S). On the basis of these data this compund may be 2,4-dimethyl-1,2 benzoquinone.

Stoichiometry of the reaction was 1 mol 2,4 dimethylaniline : 2 mol of periodate.

The data in table-1 show that the reaction is second order, being first order in each reactant, as found by the application of the Vant hoff's differential method.

A plot of k_1^{-1} vs $[S]^{-1}$ has been found to be linear with negligible intercept (Table-2) indicating the formation of an unstable intermediate.

Kinetic studies in the pH range 6.0 to 8.5 (Table-3) indicate a rate maxima at pH 7.5. The first part of this profile i.e. the increase in the rate from pH 6.0 to 7.5 may be due to the decrease in the protonation of DMA from pH 6.0 to 7.5 which makes greater concentrations of DMA available for reaction. This assumption is in line with the fact that unprotonated DMA is the reactive species as shown by us in the mechanism. Further, the concentration of the periodate monoanion is maximum around pH 5.0 to 7.0 and decreases beyond this pH value as worked out by earlier workers [1, 2, 12-14], which may be the reason for the decrease in the rate of reaction beyond pH 7.5. This behaviour also supports our assumption in the proposed mechanism (CHART) that out of the various species of the periodate the species taking part in the reaction in this case is the periodate monoanion $[IO_4^-]$.

The data in Table-4 suggests a linear relation between log k_2 vs 1/D with a negative slope indicating an ion-dipole interaction in this reaction [15]. This point was also confirmed by a high negative value of entropy of activation and no effect of the free radical scavengers like acrylamide and allyl alcohol on the reaction observed by us. A primary linear type plot between log k_2 vs ionic strength (μ) (Table-5) also indicates that the ion-dipole reaction is the rate determining step.

The values of different thermodynamic parameters as evaluated from the linear Arrhenius plot obtained from the results of kinetic studies at four temperatures ranging from 35 to 50°C are E_a = 12.436 k.cal.mol^{-1}; A = 4.274x10^{7} lit.mol^{-1}.sec.$^{-1}$; $\Delta S^{\#} = -25.723\ e.u.$; $\Delta F^{\#} = 19.926$ k.cal.mol^{-1} and $\Delta H^{\#} = 11.81$ k.cal.mol^{-1}.

On the basis of kinetic studies and the product characterized, the mechanism given in chart can be proposed which fits the rate law given below.

$$\frac{dA}{dt} = k_2[DMA][IO_4^-]$$

The formation of a charged intermediate (I) which reacts with another periodate molecule to form quinoneimine (II) and subsequent fast hydrolysis of II to give 2,4 dimethyl-1, 2 benzoquinone are are the main features of the proposed mechanism which are well supported by the product characterized by us and the different kinetic results viz. order, effect of solvent, ionic strength, free radical scavengers and pH on the rate as well as the $\Delta S^{\#}$ of this reaction.

Table-1.

λ max = 475nm, Acetone = 5.0% (V/V), Temp. = 35±0.1°C.

[DMA]x10^{3} (M)	3.0	3.0	3.0	3.0	3.0	4.0	5.0	6.0	8.0	10.0
[NaIO$_4$]x10^{3} (M)	4.0	6.0	8.0	10.0	12.0	3.0	3.0	3.0	3.0	3.0
(dA/dt)$_i$x10^{3}	6.5	9.5	13.0	7.5	20.0	5.5	7.0	8.5	11.5	14.0

Table-2.

λ max = 475nm, Acetone = 5.0% (V/V), Temp. = 35±0.1°C.

[DMA]x10^{2} (M)	1.0	1.5	2.0	2.5	3.0	0.1	0.1	0.1	0.1	0.1
[NaIO$_4$]x10^{3} (M)	0.1	0.1	0.1	0.1	0.1	1.0	1.5	2.0	2.5	3.0
k$_1$x10^{4}(min^{-1})	6.53	9.60	12.67	15.35	19.96	7.10	11.13	13.82	17.66	22.10

Table-3.

[DMA] = 1.0×10^{-2}M, $[NaIO_4] = 1.0\times10^{-3}$M, λ max = 475nm,

Acetone = 5.0% (V/V) Temp. = 35±0.1°C.

pH	6.0	6.5	7.0	7.5	8.0	8.5
$k_2\times10^2$ (lit.mol^{-1} min^{-1})	20.7	21.5	23.0	29.2	25.3	22.6

Table-4.

[DMA] = 2.0×10^{-2}M, $[NaIO_4] = 2\times10^{-3}$M, λ max = 475nm,

Temp. = 35±0.1°C.

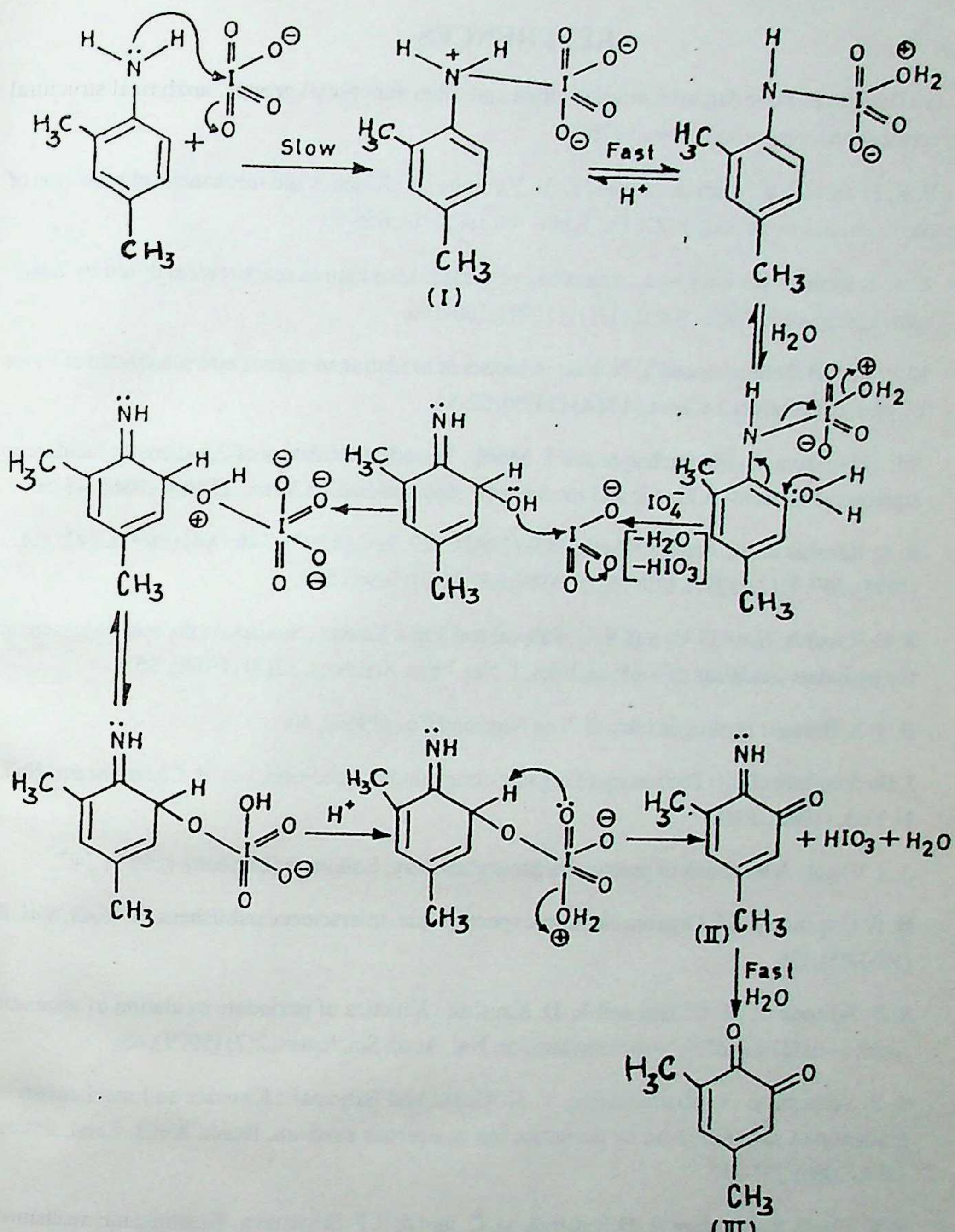
Dielectric constant (D)	72.4	70.0	66.8	64.0
1/D	138.0	142.8	150.0	156.0
$k_2\times10^2$ (lit.mol^{-1}.min^{-1})	4.415	4.030	3.455	3.070

Table-5.

[DMA] = 1.0×10^{-2}M, $[NaIO_4] = 1.0\times10^{-3}$M, λ max = 475nm,

Acetone = 5.0% (V/V)

$\mu\times10^{-3}$ (M)	2.0	4.0	6.0	9.0
$k_2\times10^2$ (lit.mol^{-1}.min^{-1})	1.77	2.23	2.74	3.99



(III)

2, 4-dimethyl-1, 2-benzoquinone

CHART

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